Exercise 5

Where does a three-dimensional wave have to vanish if its initial data ϕ and ψ vanish outside a sphere?

Solution

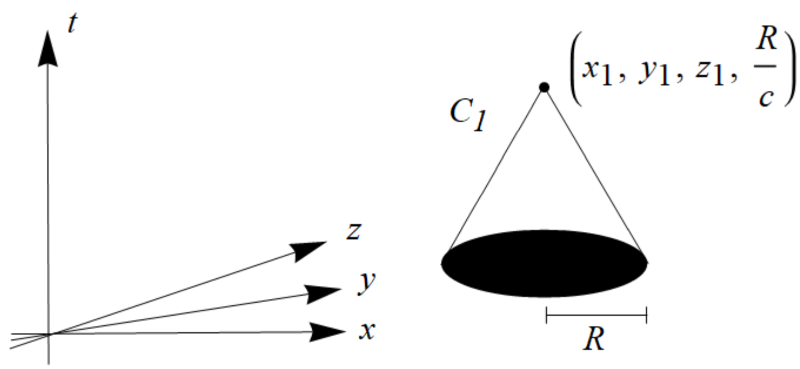
In the previous two exercises the solution to the three-dimensional wave equation subject to two initial conditions,

$$\begin{aligned} &u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y, z < \infty,\ t > 0 \\ &u(x, y, z, 0) = \phi(x, y, z) \\ &u_t(x, y, z, 0) = \psi(x, y, z), \end{aligned}$$

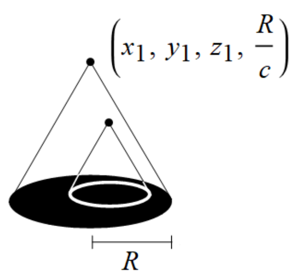
was found to be

$$u(x,y,z,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \phi(x_0,y_0,z_0)\,dS_0\right] + \frac{1}{4\pi c^2 t}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \psi(x_0,y_0,z_0)\,dS_0.$$

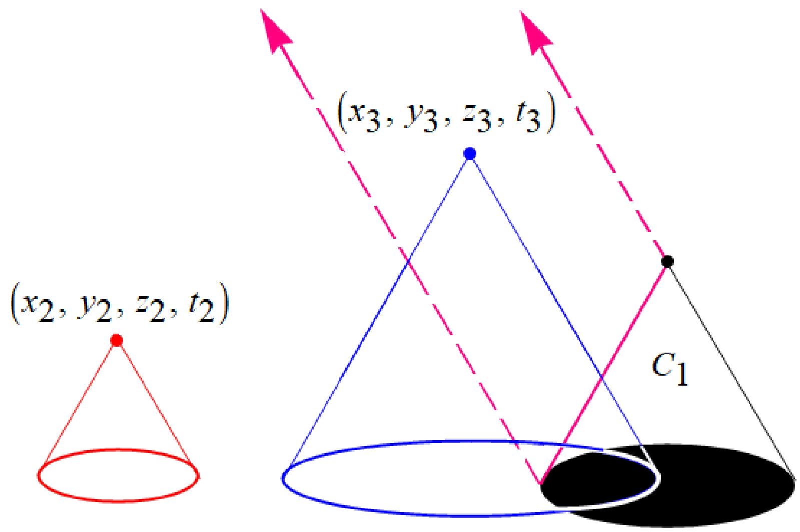
These double integrals are surface integrals over a sphere of radius ct centered at (x, y, z). Consequently, the solution for u is nonzero only if the spherical surface goes through a set of points where ϕ or ψ are nonzero. Suppose that ϕ and ψ are only nonzero within and on a sphere centered at (x_1, y_1, z_1) with radius R. This region is represented as the black hyperdisk in the figure below.



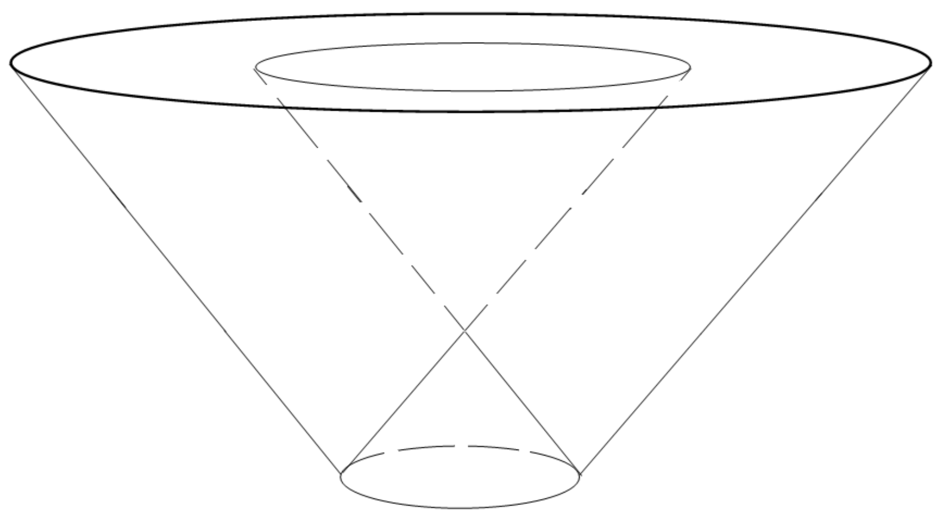
Note that because the radius is R, the apex of the cone C_1 is at $(x_1, y_1, z_1, R/c)$. At all points (x, y, z, t) within and on this cone, u is nonzero because the spherical surface of radius ct (the solid white hypercircle below) lies on points of the black hyperdisk.



Outside of C_1, u may or may not be zero. Consider the cones formed by the points, (x_2, y_2, z_2, t_2) and (x_3, y_3, z_3, t_3).



u is zero at (x_2, y_2, z_2, t_2) because the spherical surface centered at (x_2, y_2, z_2) with radius ct_2 does not lie on points of the black hyperdisk. On the other hand, u is nonzero at (x_3, y_3, z_3, t_3) because the spherical surface centered at (x_3, y_3, z_3) with radius ct_3 lies on points of the black hyperdisk, as indicated by the solid white arc. In fact, u is nonzero at any point chosen in the region enclosed by the magenta lines except where they are dashed. This situation is the same at any azimuthal angle, not just the one shown in the figure above. In other words, to obtain the general region outside C_1 where the solution is nonzero, one must rotate the region enclosed in magenta lines about the t-axis.



The union of C_1 and this new region obtained by rotation is the set of all points in space-time where the solution to the wave equation is nonzero. Geometrically, it is an inverted frustum with a conical hollow that begins a (temporal) distance R/c from the base of radius R and extends upward indefinitely. A three-dimensional wave vanishes at points outside this hollowed frustum.

In order to describe it mathematically, let A be the set of points making up the solid frustum, and let B be the set of points making up the conical hollow.

$$A = \left\{ (x, y, z, t) \,\middle|\, (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 < c^2 \left(t + \frac{R}{c} \right)^2,\ 0 < t < \infty \right\}$$
$$B = \left\{ (x, y, z, t) \,\middle|\, (x - x_1)^2 + (y - y_1)^2 + (z - z_1)^2 \le c^2 \left(t - \frac{R}{c} \right)^2,\ \frac{R}{c} < t < \infty \right\}$$

The set of points in space-time where u is nonzero is the difference between A and B.

$$A - B$$

Therefore, the set of points in space-time where u is zero is the complement of the difference between A and B.

$$(A - B)'$$