Exercise 6

(a) Let S be the sphere of center $\mathbf{x}$ and radius R. What is the surface area of $S \cap \{|\mathbf{x}| < \rho\}$, the portion of S that lies within the sphere of center $\mathbf{0}$ and radius ρ?

(b) Solve the wave equation in three dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) \equiv 0$, $\psi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, and $\psi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, where A is a constant. Sketch the regions in space-time that illustrate your answer. (This is like the hammer blow of Section 2.1.)

(c) Sketch the graph of the solution (u versus $|\mathbf{x}|$) for $t = \frac{1}{2}$, 1, and 2, assuming that $\rho = c = A = 1$. (This is a "movie" of the solution.)

(d) Sketch the graph of u versus t for $|\mathbf{x}| = \frac{1}{2}$ and 2, assuming that $\rho = c = A = 1$. (This is what a stationary observer sees.)

(e) Let $|\mathbf{x}_0| < \rho$. Ride the wave along a light ray emanating from $(\mathbf{x}_0, 0)$. That is, look at $u(\mathbf{x}_0 + t\mathbf{v}, t)$ where $|\mathbf{v}| = c$. Prove that

$$t \cdot u(\mathbf{x}_0 + t\mathbf{v}, t) \text{ converges as } t \to \infty.$$

(*Hint:* (a) Divide into cases depending on whether one sphere contains the other or not. Use the law of cosines. (b) Use Kirchhoff's formula.)

Solution

Part (a)

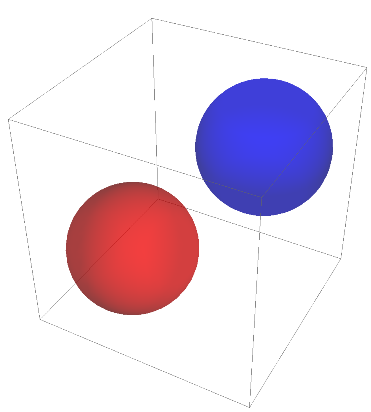
Consider two spheres in $x_0y_0z_0$-space, a red one centered at the origin with radius ρ and a blue one centered at (x, y, z) with radius R. Let r be the distance between the spheres' centers. The aim in this part is to find the surface area of the blue sphere that lies within the red sphere.

Case 1

In this case the spheres are completely separate,

$$r > \rho + R,$$

so the surface area of the blue sphere inside the red sphere is zero.

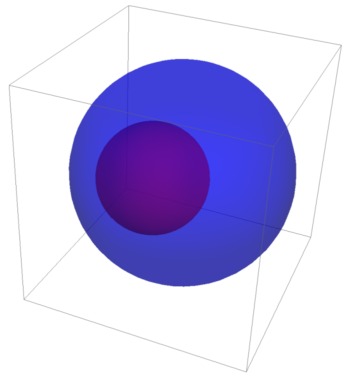


Case 2

In this case the red sphere lies completely within the blue sphere,

$$R > \rho + r,$$

so the surface area of the blue sphere inside the red sphere is zero.

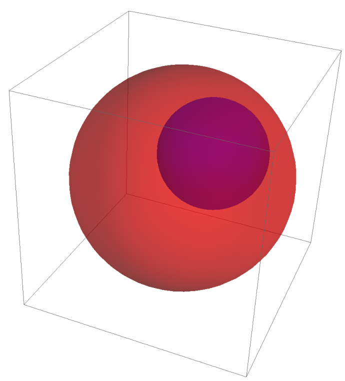


Case 3

In this case the blue sphere lies completely within the red sphere,

$$\rho > R + r,$$

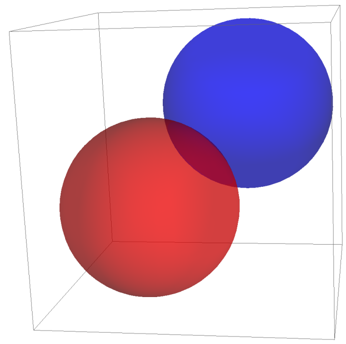
so the surface area of the blue sphere inside the red sphere is $4\pi R^2$.



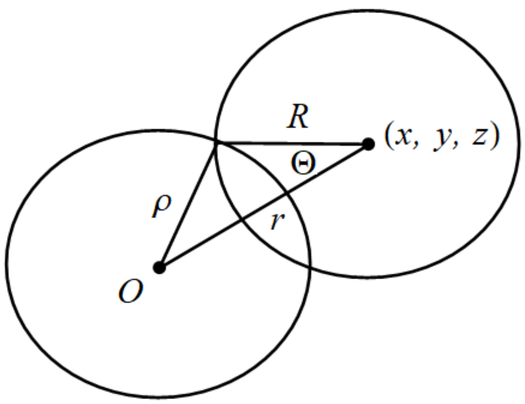
Case 4

In this case the spheres partially intersect, so $r < \rho + R$ and either $\rho < R + r$ or $R < r + \rho$, depending on whether the red sphere or the blue sphere is bigger, respectively. That is,

$$|\rho - R| < r < \rho + R.$$



In order to calculate the surface area of the portion of the blue sphere inside the red sphere, consider a new (spherical) coordinate system with its origin at (x, y, z) and polar axis pointing towards O. Let Θ be the polar angle at which the spheres touch.



The surface area is then

$$\begin{aligned} S_0 &= \iint dS_0 \\ &= \int_0^{\Theta} \int_0^{2\pi} R^2 \sin\theta_0 \, d\phi_0 \, d\theta_0 \\ &= \int_0^{\Theta} 2\pi R^2 \sin\theta_0 \, d\theta_0 \\ &= 2\pi R^2 (-\cos\theta_0) \Big|_0^{\Theta} \\ &= 2\pi R^2 (1 - \cos\Theta). \end{aligned}$$

Use the law of cosines to relate Θ with the sides of the triangle in the previous figure.

$$\rho^2 = r^2 + R^2 - 2rR\cos\Theta \quad \rightarrow \quad \cos\Theta = \frac{r^2 + R^2 - \rho^2}{2rR}$$

Substitute this expression for $\cos\Theta$ into the formula for the surface area.

$$\begin{aligned} S_0 &= 2\pi R^2 \left(1 - \frac{r^2 + R^2 - \rho^2}{2rR}\right) \\ &= 2\pi R^2 \left(\frac{2rR - r^2 - R^2 + \rho^2}{2rR}\right) \\ &= \frac{\pi R}{r}[\rho^2 - (r^2 - 2rR + R^2)] \end{aligned}$$

Therefore, the surface area of the portion of the blue sphere inside the red sphere is

$$S_0 = \frac{\pi R}{r}[\rho^2 - (r - R)^2].$$

Part (b)

The solution of the three-dimensional wave equation in space subject to two initial conditions,

$$\begin{aligned} &u_{tt} = c^2 \nabla^2 u, \quad -\infty < x, y, z < \infty,\ t > 0 \\ &u(x, y, z, 0) = \alpha(x, y, z) \\ &u_t(x, y, z, 0) = \beta(x, y, z), \end{aligned}$$

is given by the formula of Kirchhoff and Poisson.

$$u(x, y, z, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \alpha(x_0, y_0, z_0)\, dS_0\right] + \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \beta(x_0, y_0, z_0)\, dS_0$$

In particular, we wish to solve the initial value problem when

$$\alpha(x, y, z) = 0 \quad \text{and} \quad \beta(x, y, z) = \begin{cases} A & r < \rho \\ 0 & r > \rho \end{cases}.$$

With these initial conditions, the previous formula simplifies to

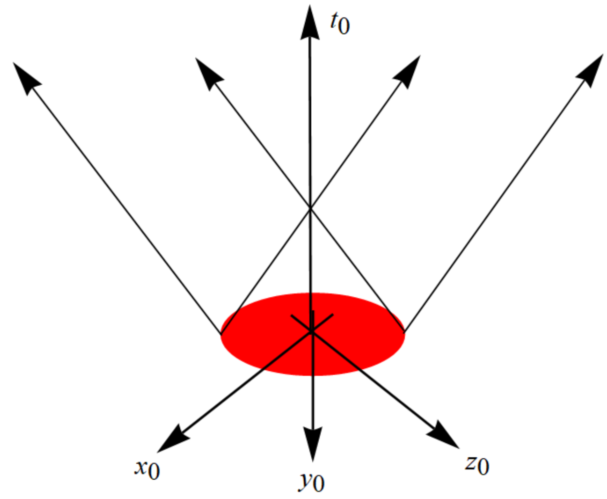
$$u(x, y, z, t) = \frac{1}{4\pi c^2 t} \iint\limits_{Q \cap T} A\, dS_0,$$

where

$$\begin{aligned} Q &= \{(x_0, y_0, z_0) \mid (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = c^2 t^2\} \\ T &= \{(x_0, y_0, z_0) \mid x_0^2 + y_0^2 + z_0^2 < \rho^2\}. \end{aligned}$$

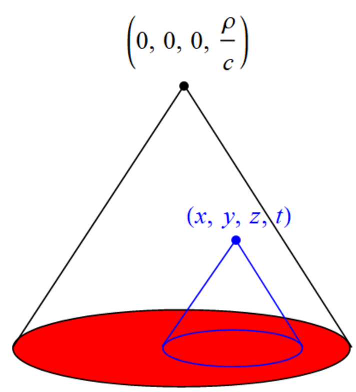
Basically, this surface integral is over the part of the sphere centered at (x, y, z) with radius ct that lies within the solid ball centered at the origin with radius ρ. Depending what region in space-time the point (x, y, z, t) is chosen, the surface integral will yield a different result.

The characteristic surfaces, which are obtained by drawing light rays (with slope c) from every point on the boundary of the hyperdisk within which the initial condition is nonzero, separate these regions.



The red hyperdisk represents the solid ball in $x_0y_0z_0$-space where the initial condition is nonzero. Because it has radius ρ, the height of the cone formed by the crossing characteristic lines is ρ/c. u will be calculated within this cone first.

The First Region

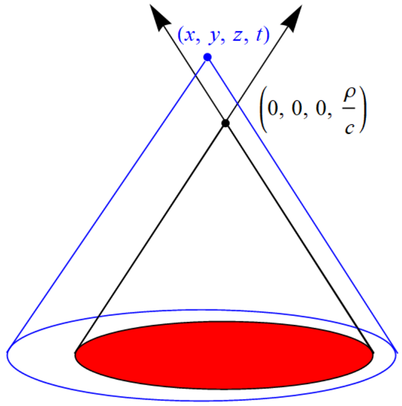


The blue circle illustrated above represents the sphere centered at (x, y, z) with radius ct. Since it lies within the red disk, the surface area of the blue sphere that lies within the red ball is $4\pi(ct)^2$. Consequently,

$$\begin{aligned} u(x, y, z, t) &= \frac{1}{4\pi c^2 t} \iint_{Q\cap T} A \, dS_0 \\ &= \frac{A}{4\pi c^2 t}(4\pi c^2 t^2) \\ &= At. \end{aligned}$$

This formula for u is valid at points in space-time where $\rho > ct + r$, or $r < \rho - ct$. u will now be calculated in the region directly above the cone.

The Second Region

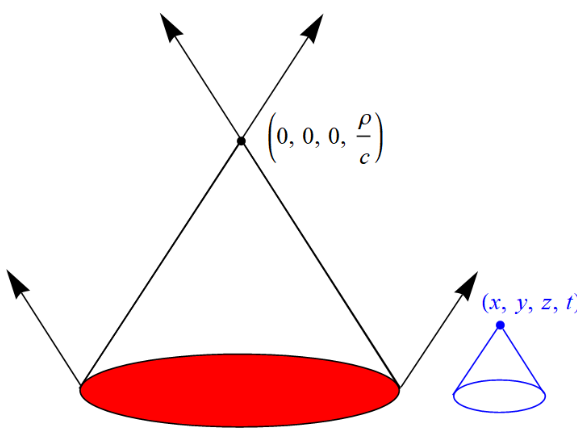


The blue circle illustrated above represents the sphere centered at (x, y, z) with radius ct. Since the red disk lies within it, the surface area of the blue sphere that lies within the red ball is zero. Consequently,

$$\begin{aligned} u(x, y, z, t) &= \frac{1}{4\pi c^2 t} \iint_{Q \cap T} A \, dS_0 \\ &= \frac{A}{4\pi c^2 t}(0) \\ &= 0. \end{aligned}$$

This formula for u is valid at points in space-time where $ct > \rho + r$, or $r < ct - \rho$. u will now be calculated in the region outside the cone right above the $x_0y_0z_0$-plane.

The Third Region

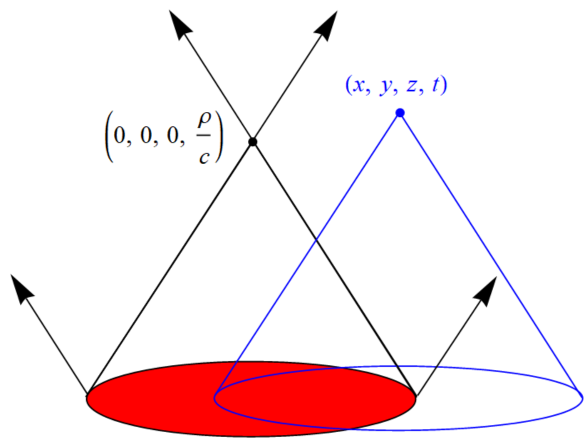


The blue circle illustrated above represents the sphere centered at (x, y, z) with radius ct. Since it and the red disk are completely separate, the surface area of the blue sphere that lies within the red ball is zero. Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{1}{4\pi c^2 t} \iint_{Q\cap T} A\, dS_0 \\ &= \frac{A}{4\pi c^2 t}(0) \\ &= 0. \end{aligned}$$

This formula for u is valid at points in space-time where $r > \rho + ct$. u will now be calculated in the last region outside the cone.

<u>The Fourth Region</u>



The blue circle illustrated above represents the sphere centered at (x, y, z) with radius ct. Since it partially intersects the red disk, the surface area of the portion of the blue sphere that lies within the red ball is

$$\frac{\pi(ct)}{r}[\rho^2 - (r - ct)^2].$$

Consequently,

$$\begin{aligned} u(x,y,z,t) &= \frac{1}{4\pi c^2 t} \iint_{Q\cap T} A\, dS_0 \\ &= \frac{A}{4\pi c^2 t}\frac{\pi(ct)}{r}[\rho^2 - (r - ct)^2] \\ &= \frac{A}{4cr}[\rho^2 - (r - ct)^2]. \end{aligned}$$

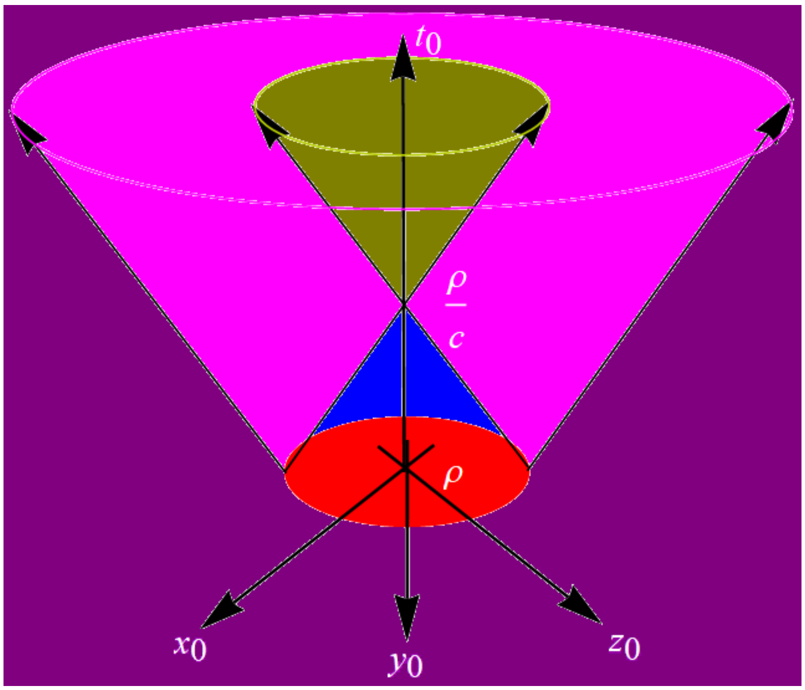
This formula for u is valid at points in space-time where $|\rho - ct| < r < \rho + ct$. To summarize the results, we have

$$u(x,y,z,t) = \begin{cases} At & \text{if } r < \rho - ct \\ 0 & \text{if } r < ct - \rho \\ 0 & \text{if } r > \rho + ct \\ \frac{A}{4cr}[\rho^2 - (r - ct)^2] & \text{if } |\rho - ct| < r < \rho + ct \end{cases}.$$

Note that substituting $\rho - ct$, $ct - \rho$, and $\rho + ct$ for r in the magenta solution results in the blue, olive, and purple solutions, respectively. In other words, u is continuous across each region, so each of the $<$ and $>$ signs can be changed to $\leq$ and $\geq$. Therefore, replacing r with $\sqrt{x^2+y^2+z^2}$,

$$u(x,y,z,t) = \begin{cases} At & \text{if } \sqrt{x^2+y^2+z^2} \leq \rho - ct \\ 0 & \text{if } \sqrt{x^2+y^2+z^2} \leq ct - \rho \\ 0 & \text{if } \sqrt{x^2+y^2+z^2} \geq \rho + ct \\ \dfrac{A}{4c\sqrt{x^2+y^2+z^2}}\left[\rho^2 - (\sqrt{x^2+y^2+z^2} - ct)^2\right] & \text{if } |\rho - ct| \leq \sqrt{x^2+y^2+z^2} \leq \rho + ct \end{cases}.$$

Space-time is illustrated below; the solution to the wave equation in each region is labeled by color.



Part (c)

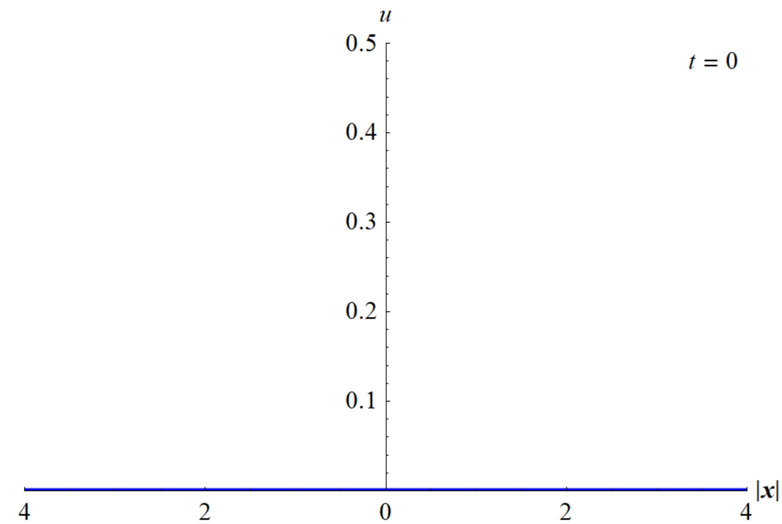
Set $\rho = c = A = 1$ and replace $\sqrt{x^2+y^2+z^2}$ with $|\mathbf{x}|$ in the solution.

$$u(x,y,z,t) = \begin{cases} t & \text{if } |\mathbf{x}| < 1 - t \\ 0 & \text{if } |\mathbf{x}| < t - 1 \\ 0 & \text{if } |\mathbf{x}| > 1 + t \\ \dfrac{1}{4|\mathbf{x}|}\left[1 - (|\mathbf{x}| - t)^2\right] & \text{if } |1 - t| < |\mathbf{x}| < 1 + t \end{cases}$$

u is only a function of $|\mathbf{x}|$ and t, so $u = u(|\mathbf{x}|, t)$.

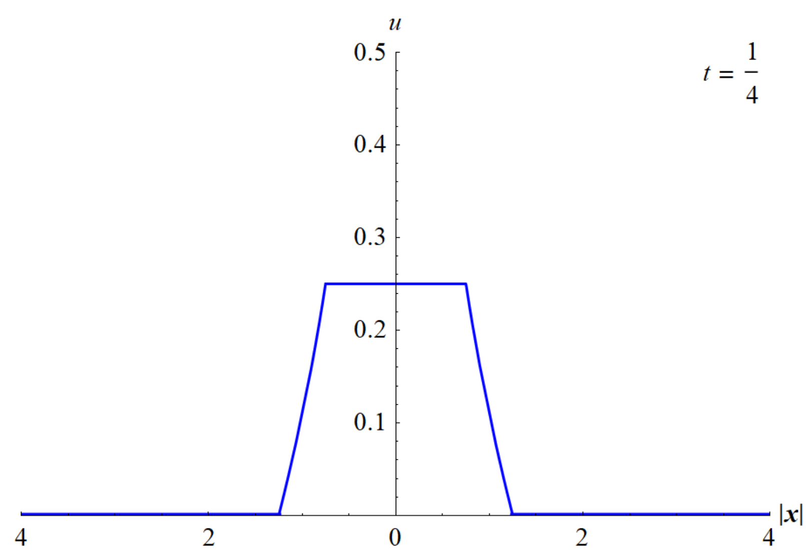
If $t = 0$, then

$$u(|\mathbf{x}|, 0) = \begin{cases} 0 & \text{if } |\mathbf{x}| < 1 \\ 0 & \text{if } |\mathbf{x}| < -1 \\ 0 & \text{if } |\mathbf{x}| > 1 \\ \dfrac{1}{4|\mathbf{x}|} - \dfrac{|\mathbf{x}|}{4} & \text{if } 1 < |\mathbf{x}| < 1 \end{cases}.$$



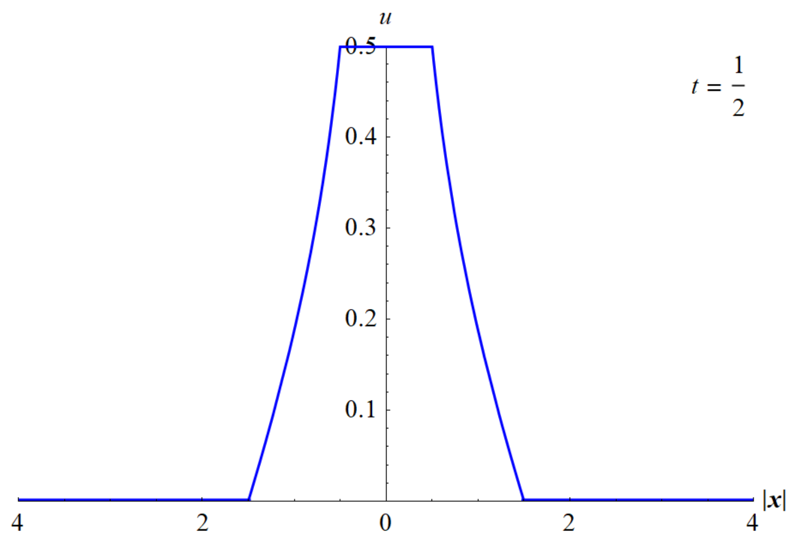
If $t = \frac{1}{4}$, then

$$u\left(|\mathbf{x}|, \frac{1}{4}\right) = \begin{cases} \dfrac{1}{4} & \text{if } |\mathbf{x}| < \dfrac{3}{4} \\ 0 & \text{if } |\mathbf{x}| < -\dfrac{3}{4} \\ 0 & \text{if } |\mathbf{x}| > \dfrac{5}{4} \\ \dfrac{1}{8} + \dfrac{15}{64|\mathbf{x}|} - \dfrac{|\mathbf{x}|}{4} & \text{if } \dfrac{3}{4} < |\mathbf{x}| < \dfrac{5}{4} \end{cases}.$$



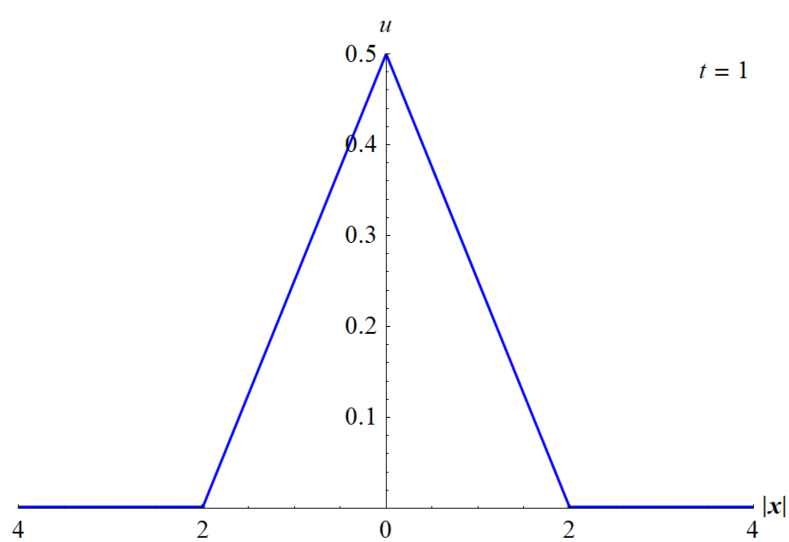
If $t = \frac{1}{2}$, then

$$u\left(|\mathbf{x}|, \frac{1}{2}\right) = \begin{cases} \frac{1}{2} & \text{if } |\mathbf{x}| < \frac{1}{2} \\ 0 & \text{if } |\mathbf{x}| < -\frac{1}{2} \\ 0 & \text{if } |\mathbf{x}| > \frac{3}{2} \\ \frac{1}{4} + \frac{3}{16|\mathbf{x}|} - \frac{|\mathbf{x}|}{4} & \text{if } \frac{1}{2} < |\mathbf{x}| < \frac{3}{2} \end{cases}.$$



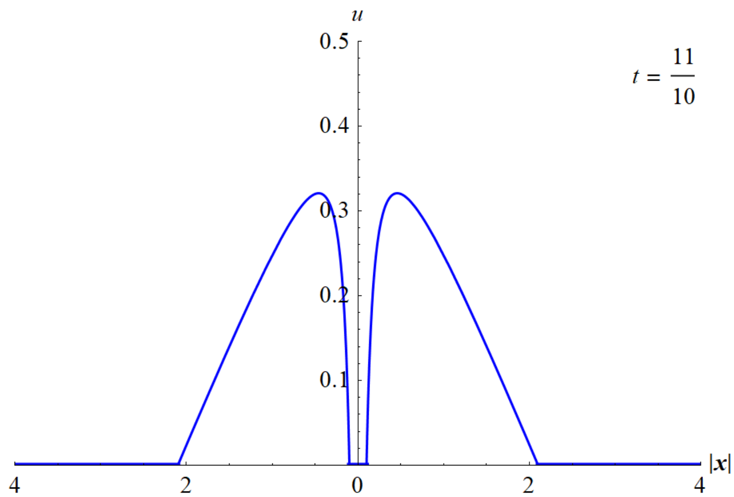
If $t = 1$, then

$$u(|\mathbf{x}|, 1) = \begin{cases} 1 & \text{if } |\mathbf{x}| < 0 \\ 0 & \text{if } |\mathbf{x}| < 0 \\ 0 & \text{if } |\mathbf{x}| > 2 \\ \frac{1}{2} - \frac{|\mathbf{x}|}{4} & \text{if } 0 < |\mathbf{x}| < 2 \end{cases}.$$



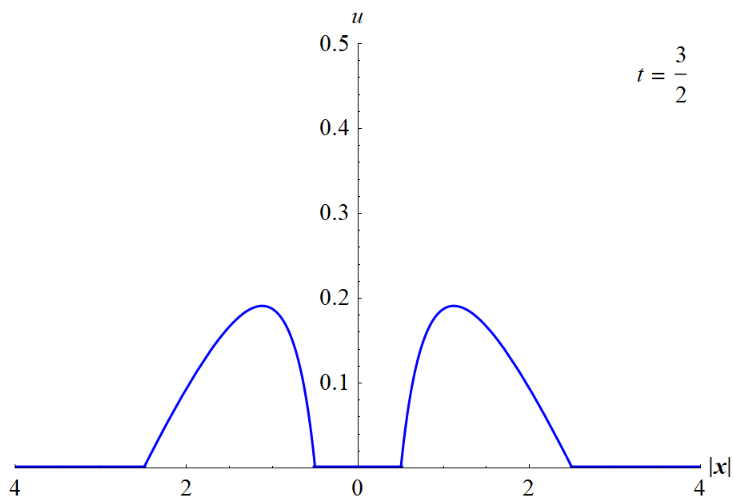
If $t = \frac{11}{10}$, then

$$u\left(|\mathbf{x}|, \frac{11}{10}\right) = \begin{cases} \frac{11}{10} & \text{if } |\mathbf{x}| < -\frac{1}{10} \\ 0 & \text{if } |\mathbf{x}| < \frac{1}{10} \\ 0 & \text{if } |\mathbf{x}| > \frac{21}{10} \\ \frac{11}{20} - \frac{21}{400|\mathbf{x}|} - \frac{|\mathbf{x}|}{4} & \text{if } \frac{1}{10} < |\mathbf{x}| < \frac{21}{10} \end{cases}.$$



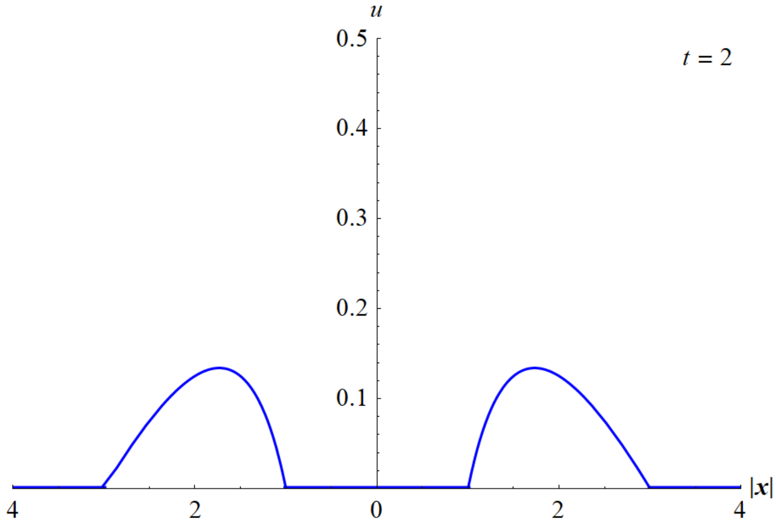
If $t = \frac{3}{2}$, then

$$u\left(|\mathbf{x}|, \frac{3}{2}\right) = \begin{cases} \frac{3}{2} & \text{if } |\mathbf{x}| < -\frac{1}{2} \\ 0 & \text{if } |\mathbf{x}| < \frac{1}{2} \\ 0 & \text{if } |\mathbf{x}| > \frac{5}{2} \\ \frac{3}{4} - \frac{5}{16|\mathbf{x}|} - \frac{|\mathbf{x}|}{4} & \text{if } \frac{1}{2} < |\mathbf{x}| < \frac{5}{2} \end{cases}.$$



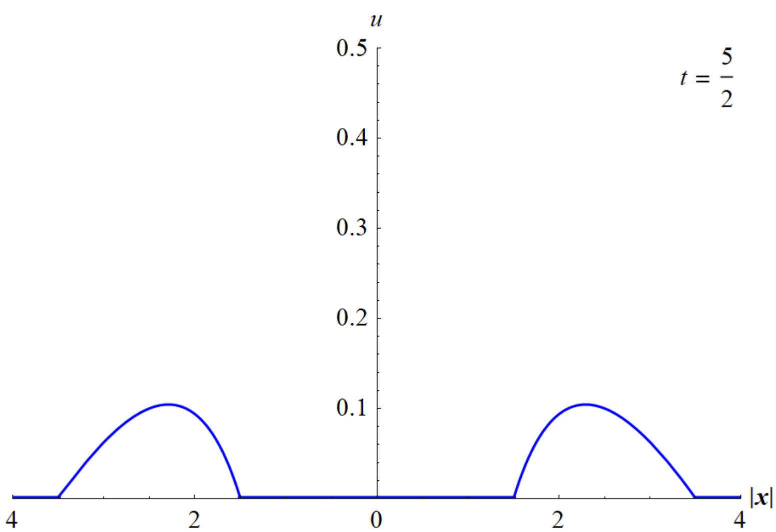
If $t = 2$, then

$$u(|\mathbf{x}|, 2) = \begin{cases} 2 & \text{if } |\mathbf{x}| < -1 \\ 0 & \text{if } |\mathbf{x}| < 1 \\ 0 & \text{if } |\mathbf{x}| > 3 \\ 1 - \dfrac{3}{4|\mathbf{x}|} - \dfrac{|\mathbf{x}|}{4} & \text{if } 1 < |\mathbf{x}| < 3 \end{cases}.$$



If $t = \frac{5}{2}$, then

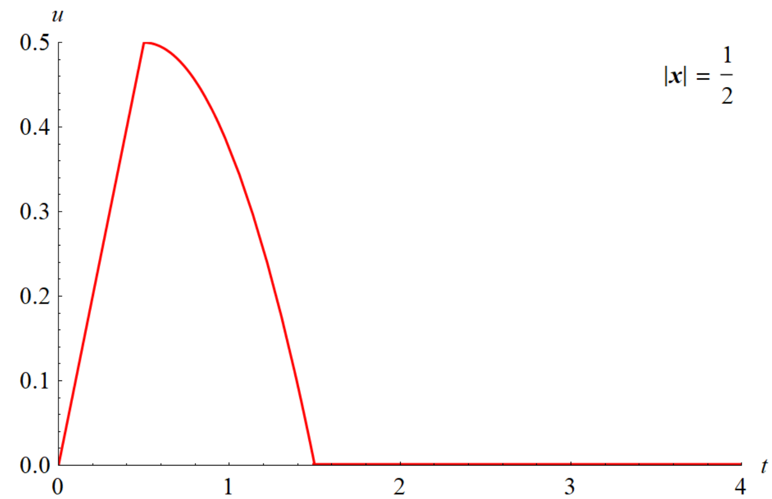
$$u\left(|\mathbf{x}|, \frac{5}{2}\right) = \begin{cases} \dfrac{5}{2} & \text{if } |\mathbf{x}| < -\dfrac{3}{2} \\ 0 & \text{if } |\mathbf{x}| < \dfrac{3}{2} \\ 0 & \text{if } |\mathbf{x}| > \dfrac{7}{2} \\ \dfrac{5}{4} - \dfrac{21}{16|\mathbf{x}|} - \dfrac{|\mathbf{x}|}{4} & \text{if } \dfrac{3}{2} < |\mathbf{x}| < \dfrac{7}{2} \end{cases}.$$



Part (d)

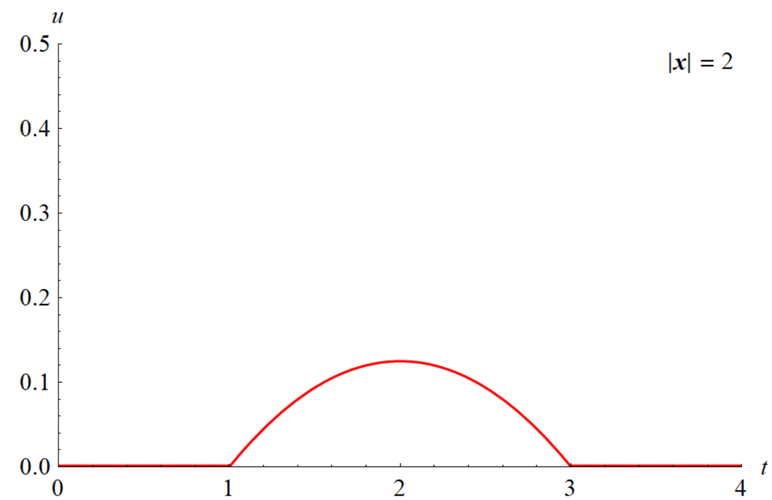
If $|\mathbf{x}| = \frac{1}{2}$, then

$$u\left(\frac{1}{2}, t\right) = \begin{cases} t & \text{if } t < \dfrac{1}{2} \\ 0 & \text{if } t > \dfrac{3}{2} \\ 0 & \text{if } t < -\dfrac{1}{2} \\ \dfrac{3}{8} + \dfrac{t}{2} - \dfrac{t^2}{2} & \text{if } \dfrac{1}{2} < t < \dfrac{3}{2} \end{cases}.$$



If $|\mathbf{x}| = 2$, then

$$u(2, t) = \begin{cases} t & \text{if } t < -1 \\ 0 & \text{if } t > 3 \\ 0 & \text{if } t < 1 \\ -\dfrac{3}{8} + \dfrac{t}{2} - \dfrac{t^2}{8} & \text{if } 1 < t < 3 \end{cases}.$$



Part (e)

Let $\mathbf{x}_1$ be a point in space-time within the red hyperdisk initially: $|\mathbf{x}_1| < \rho$ when $t = 0$. The aim here is to show that

$$\lim_{t\to\infty} t \cdot u(\mathbf{x}_1 + t\mathbf{v}, t)$$

converges, where $|\mathbf{v}| = c$. Replace $\sqrt{x^2+y^2+z^2}$ with $|\mathbf{x}|$ and (x, y, z) with $\mathbf{x}$ in the solution for u.

$$u(\mathbf{x}, t) = \begin{cases} At & \text{if } |\mathbf{x}| < \rho - ct \\ 0 & \text{if } |\mathbf{x}| < ct - \rho \\ 0 & \text{if } |\mathbf{x}| > \rho + ct \\ \dfrac{A}{4c|\mathbf{x}|}\left[\rho^2 - (|\mathbf{x}| - ct)^2\right] & \text{if } |\rho - ct| < |\mathbf{x}| < \rho + ct \end{cases}$$

$(\mathbf{x}_1 + t\mathbf{v}, t)$ lies within the magenta region for large t, so

$$\begin{aligned}
\lim_{t\to\infty} t \cdot u(\mathbf{x}_1 + t\mathbf{v}, t) &= \lim_{t\to\infty} t \cdot \frac{A}{4c|\mathbf{x}_1 + t\mathbf{v}|}\left[\rho^2 - (|\mathbf{x}_1 + t\mathbf{v}| - ct)^2\right] \\
&= \lim_{t\to\infty} \frac{At}{4c\sqrt{(\mathbf{x}_1 + t\mathbf{v})^2}}\left[\rho^2 - \left(\sqrt{(\mathbf{x}_1 + t\mathbf{v})^2} - ct\right)^2\right] \\
&= \lim_{t\to\infty} \frac{At}{4c\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + t^2|\mathbf{v}|^2}}\left[\rho^2 - \left(\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + t^2|\mathbf{v}|^2} - ct\right)^2\right] \\
&= \lim_{t\to\infty} \frac{At}{4c\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + c^2t^2}}\left[\rho^2 - \left(\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + c^2t^2} - ct\right)^2\right] \\
&= \lim_{t\to\infty} \frac{A}{\frac{4c}{t}\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + c^2t^2}}\left[\rho^2 - c^2t^2\left(\frac{1}{ct}\sqrt{|\mathbf{x}_1|^2 + 2t\mathbf{x}_1\cdot\mathbf{v} + c^2t^2} - 1\right)^2\right] \\
&= \lim_{t\to\infty} \frac{A}{4c\sqrt{\frac{|\mathbf{x}_1|^2}{t^2} + \frac{2\mathbf{x}_1\cdot\mathbf{v}}{t} + c^2}}\left[\rho^2 - c^2t^2\left(\sqrt{\frac{|\mathbf{x}_1|^2}{c^2t^2} + \frac{2\mathbf{x}_1\cdot\mathbf{v}}{c^2t} + 1} - 1\right)^2\right] \\
&= \left(\lim_{t\to\infty} \frac{A}{4c\sqrt{\cancelto{0}{\frac{|\mathbf{x}_1|^2}{t^2}} + \cancelto{0}{\frac{2\mathbf{x}_1\cdot\mathbf{v}}{t}} + c^2}}\right)\lim_{t\to\infty}\left[\rho^2 - c^2t^2\left(\sqrt{1 + \frac{2\mathbf{x}_1\cdot\mathbf{v}}{c^2t} + \frac{|\mathbf{x}_1|^2}{c^2t^2}} - 1\right)^2\right].
\end{aligned}$$

Use the binomial series for the square root in the second limit.

$$\begin{aligned}
&= \left(\frac{A}{4c^2}\right)\lim_{t\to\infty}\left\{\rho^2 - c^2t^2\left[1 + \frac{\mathbf{x}_1\cdot\mathbf{v}}{c^2t} + \frac{|\mathbf{x}_1|^2}{2c^2t^2} + O\left(\frac{1}{t^2}\right) - 1\right]^2\right\} \\
&= \frac{A}{4c^2}\lim_{t\to\infty}\left\{\rho^2 - \left[\frac{\mathbf{x}_1\cdot\mathbf{v}}{c} + \cancelto{0}{\frac{|\mathbf{x}_1|^2}{2ct}} + \cancelto{0}{O\left(\frac{1}{t}\right)}\right]^2\right\} \\
&= \frac{A}{4c^2}\left[\rho^2 - \frac{(\mathbf{x}_1\cdot\mathbf{v})^2}{c^2}\right]
\end{aligned}$$

Therefore, $t \cdot u(\mathbf{x}_1 + t\mathbf{v}, t)$ converges as $t \to \infty$.