Exercise 6

(a) Let S be the sphere of center $\mathbf{x}$ and radius R. What is the surface area of $S \cap \{|\mathbf{x}| < \rho\}$, the portion of S that lies within the sphere of center $\mathbf{0}$ and radius ρ?

(b) Solve the wave equation in three dimensions for $t > 0$ with the initial conditions $\phi(\mathbf{x}) \equiv 0$, $\psi(\mathbf{x}) = A$ for $|\mathbf{x}| < \rho$, and $\psi(\mathbf{x}) = 0$ for $|\mathbf{x}| > \rho$, where A is a constant. Sketch the regions in space-time that illustrate your answer. (This is like the hammer blow of Section 2.1.)

(c) Sketch the graph of the solution (u versus $|\mathbf{x}|$) for $t = \frac{1}{2}$, 1, and 2, assuming that $\rho = c = A = 1$. (This is a "movie" of the solution.)

(d) Sketch the graph of u versus t for $|\mathbf{x}| = \frac{1}{2}$ and 2, assuming that $\rho = c = A = 1$. (This is what a stationary observer sees.)

(e) Let $|\mathbf{x}_0| < \rho$. Ride the wave along a light ray emanating from $(\mathbf{x}_0, 0)$. That is, look at $u(\mathbf{x}_0 + t\mathbf{v}, t)$ where $|\mathbf{v}| = c$. Prove that

$$t \cdot u(\mathbf{x}_0 + t\mathbf{v}, t) \text{ converges as } t \to \infty.$$

(*Hint:* (a) Divide into cases depending on whether one sphere contains the other or not. Use the law of cosines. (b) Use Kirchhoff's formula.)