Exercise 1

Prove that $\Delta(\overline{u}) = (\overline{\Delta u})$ for any function; that is, the laplacian of the average is the average of the laplacian. (Hint: Write $\Delta u$ in spherical coordinates and show that the angular terms have zero average on spheres centered at the origin.)

Solution

$\overline{u} = \overline{u}(r, t)$ is defined to be the average of $u$ over a sphere of radius $r$,

$$
\overline{u} = \frac{1}{4\pi r^2} \int_0^{2\pi} \int_0^\pi u(r, \phi, \theta, t) r^2 \sin \theta \, d\theta \, d\phi
$$

and the Laplacian operator expands in spherical coordinates as

$$
\Delta = \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2},
$$

where $\theta$ is the angle from the polar axis. Note that $\overline{u}$ is only a function of $r$ and $t$ because the double integral over the surface wipes out the $\phi$ and $\theta$ variables. We have

$$
\Delta(\overline{u}) = \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \right) \overline{u}
$$

$$
= \frac{\partial^2}{\partial r^2} \overline{u} + \frac{2}{r} \frac{\partial}{\partial r} \overline{u} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \overline{u} + \frac{\cot \theta}{r^2} \frac{\partial}{\partial \theta} \overline{u} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2}{\partial \phi^2} \overline{u}
$$

$$
= \frac{\partial^2}{\partial r^2} \left[ \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi u(r, \phi', \theta', t) \sin \theta' \, d\theta' \, d\phi' \right] + \frac{2}{r} \frac{\partial}{\partial r} \left[ \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi u(r, \phi', \theta', t) \sin \theta' \, d\theta' \, d\phi' \right]
$$

$$
= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{\partial^2 u}{\partial r^2} \sin \theta' \, d\theta' \, d\phi' + \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \frac{2}{r} \frac{\partial u}{\partial r} \sin \theta' \, d\theta' \, d\phi'
$$

$$
= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \left( \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right) \sin \theta' \, d\theta' \, d\phi'.
$$

Primes were put on the dummy integration variables to distinguish them from the spatial variables. Now we will show that $\overline{\Delta u}$ yields the same expression.

$$
\overline{\Delta u} = \frac{\iint \Delta u \, dS}{\iint dS} = \frac{\int_0^{2\pi} \int_0^\pi \Delta u \, r^2 \sin \theta \, d\theta \, d\phi}{4\pi r^2}
$$

$$
= \frac{1}{4\pi} \int_0^{2\pi} \int_0^\pi \Delta u \sin \theta \, d\theta \, d\phi
$$
\[ \Delta u = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left( \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \sin \theta \, d\theta \, d\phi \]

= \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left( \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right) \sin \theta \, d\theta \, d\phi

+ \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left( \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \sin \theta \, d\theta \, d\phi

All that’s left to do is to show that this second double integral vanishes.

\[ \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left( \frac{1}{r^2} \frac{\partial^2 u}{\partial \theta^2} + \frac{\cot \theta}{r^2} \frac{\partial u}{\partial \theta} + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \sin \theta \, d\theta \, d\phi \]

= \frac{1}{4\pi r^2} \int_0^{2\pi} \int_0^{\pi} \left( \sin \theta \frac{\partial^2 u}{\partial \theta^2} + \cos \theta \frac{\partial u}{\partial \theta} + \frac{1}{\sin \theta} \frac{\partial^2 u}{\partial \phi^2} \right) \, d\theta \, d\phi

= \frac{1}{4\pi r^2} \int_0^{2\pi} \int_0^{\pi} \left[ \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) + \frac{1}{\sin \theta} \frac{\partial^2 u}{\partial \phi^2} \right] \, d\theta \, d\phi

= \frac{1}{4\pi r^2} \left[ \int_0^{2\pi} \frac{1}{\sin \theta} \int_0^{\pi} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) \, d\theta \, d\phi + \int_0^{2\pi} \frac{1}{\sin \theta} \int_0^{\pi} \frac{\partial^2 u}{\partial \phi^2} \, d\theta \, d\phi \right]

= \frac{1}{4\pi r^2} \left[ \int_0^{2\pi} \left( \sin \theta \frac{\partial u}{\partial \theta} \right) \bigg|_{\theta=0}^{\theta=\pi} \, d\phi \right]

= 0

The last step follows because \( \frac{\partial u}{\partial \phi} \) has the same value at \( \phi = 0 \) as it does at \( \phi = 2\pi \). As a result,

\[ \Delta u = \frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} \left( \frac{\partial^2 u}{\partial r^2} + \frac{2}{r} \frac{\partial u}{\partial r} \right) \sin \theta \, d\theta \, d\phi \]

Therefore,

\[ \Delta(\bar{u}) = (\Delta u) \]