Exercise 4

Solve the wave equation in three dimensions with the initial data $\phi \equiv 0$, $\psi(x, y, z) = x^2 + y^2 + z^2$. (*Hint:* Use (5).)

Solution

Equation (3) is the solution to the three-dimensional wave equation in space subject to two initial conditions.

$$\begin{aligned}
&u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y, z < \infty,\ t > 0 \\
&u(x, y, z, 0) = \alpha(x, y, z) \\
&u_t(x, y, z, 0) = \beta(x, y, z)
\end{aligned}$$

Integrate both sides of the wave equation over the black hyperdisk shown in the figure below.

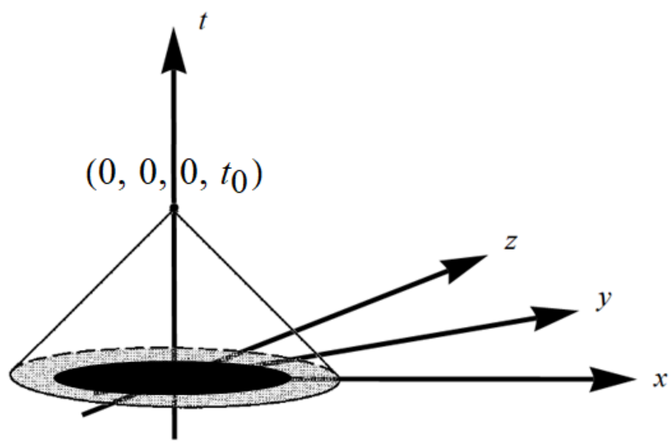


Figure 1: The black hyperdisk lies in the xyz-plane, has center $(0, 0, 0)$ and radius r, and represents a solid ball in xyz-space. Note that the shaded hyperdisk it lies within has radius ct_0, where t_0 is a particular time we want to evaluate u at.

$$\iiint_V u_{tt}\, dV = \iiint_V c^2\nabla^2 u\, dV$$

$$\iiint_V u_{tt}\, dV = c^2 \iiint_V \nabla \cdot \nabla u\, dV$$

Apply the divergence theorem to the volume integral on the right side to turn it into a surface integral over the solid ball's boundary.

$$\iiint_V u_{tt}\, dV = c^2 \oiint_S \nabla u \cdot \hat{\mathbf{n}}\, dS$$

The unit vector normal to the boundary is the radial unit vector: $\hat{\mathbf{n}} = \hat{\mathbf{r}}$. $\nabla u \cdot \hat{\mathbf{r}}$ can be interpreted as the directional derivative in the radial direction, that is, $\partial u/\partial r$.

$$\iiint_V \frac{\partial^2 u}{\partial t^2}\, dV = c^2 \oiint_S \frac{\partial u}{\partial r}\, dS$$

Write out the volume and surface integrals explicitly by using spherical coordinates (r, ϕ, θ). Here θ denotes the angle from the polar axis.

$$\int_0^\pi \int_0^{2\pi} \int_0^r \frac{\partial^2 u}{\partial t^2}(\rho^2 \sin\theta \, d\rho \, d\phi \, d\theta) = c^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r}(r^2 \sin\theta \, d\phi \, d\theta)$$

$$\int_0^r \rho^2 \int_0^\pi \int_0^{2\pi} \frac{\partial^2 u}{\partial t^2} \sin\theta \, d\phi \, d\theta \, d\rho = c^2 r^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r} \sin\theta \, d\phi \, d\theta$$

$$\int_0^r \rho^2 \frac{\partial^2}{\partial t^2}\left[\int_0^\pi \int_0^{2\pi} u(\rho, \phi, \theta, t) \sin\theta \, d\phi \, d\theta\right] d\rho = c^2 r^2 \frac{\partial}{\partial r}\left[\int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin\theta \, d\phi \, d\theta\right]$$

Let

$$v(r, t) = \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin\theta \, d\phi \, d\theta$$

so that the previous equation becomes

$$\int_0^r \rho^2 \frac{\partial^2 v}{\partial t^2} \, d\rho = c^2 r^2 \frac{\partial v}{\partial r}.$$

Differentiate both sides with respect to r to eliminate the integral on the left side.

$$r^2 \frac{\partial^2 v}{\partial t^2} = c^2 \left(2r \frac{\partial v}{\partial r} + r^2 \frac{\partial^2 v}{\partial r^2}\right)$$

Divide both sides by r.

$$r \frac{\partial^2 v}{\partial t^2} = c^2 \left(2 \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2}\right)$$

Now make the change of variables $w(r, t) = rv(r, t)$. Write the new derivatives in terms of the old ones by differentiating this substitution.

$$\begin{aligned}
\frac{\partial^2 w}{\partial t^2} &= r \frac{\partial^2 v}{\partial t^2} \\
\frac{\partial w}{\partial r} &= v + r \frac{\partial v}{\partial r} \\
\frac{\partial^2 w}{\partial r^2} &= 2 \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2}
\end{aligned}$$

The transformed PDE is the wave equation on a semi-infinite interval

$$\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial r^2}, \quad 0 < r < \infty, \ t > 0$$

subject to the Dirichlet boundary condition $w(0, t) = 0$ and the initial conditions,

$$\begin{aligned}
w(r, 0) &= r \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, 0) \sin\theta \, d\phi \, d\theta & \text{and} \quad w_t(r, 0) &= r \int_0^\pi \int_0^{2\pi} u_t(r, \phi, \theta, 0) \sin\theta \, d\phi \, d\theta \\
&= r \int_0^\pi \int_0^{2\pi} \alpha(r, \phi, \theta) \sin\theta \, d\phi \, d\theta & &= r \int_0^\pi \int_0^{2\pi} \beta(r, \phi, \theta) \sin\theta \, d\phi \, d\theta.
\end{aligned}$$

The method of reflection can be applied to solve this one-dimensional wave equation. Consider the corresponding problem over the whole line,

$$\begin{gathered} W_{tt} = c^2 W_{rr}, \quad -\infty < r < \infty,\ t > 0 \\ W(r,0) = A_{\text{odd}}(r), \quad W_t(r,0) = B_{\text{odd}}(r), \end{gathered}$$

where the odd extensions of the initial conditions for w, $A_{\text{odd}}(r)$ and $B_{\text{odd}}(r)$, are used in order to satisfy the Dirichlet boundary condition.

$$A_{\text{odd}}(r) = \begin{cases} r\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(r,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r > 0 \\ -(-r)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(-r,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r < 0 \end{cases}$$

$$B_{\text{odd}}(r) = \begin{cases} r\displaystyle\int_0^\pi \int_0^{2\pi} \beta(r,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r > 0 \\ -(-r)\displaystyle\int_0^\pi \int_0^{2\pi} \beta(-r,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r < 0 \end{cases}$$

The solution for W is given by d'Alembert's formula in section 2.1 on page 36.

$$W(r,t) = \frac{1}{2}[A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} B_{\text{odd}}(s)\, ds$$

The solution for w is then just the restriction of W to $r > 0$.

$$w(r,t) = \frac{1}{2}[A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} B_{\text{odd}}(s)\, ds, \quad r > 0$$

Our task now is to write this formula in terms of the given functions, α and β. Note that

$$A_{\text{odd}}(r+ct) = \begin{cases} (r+ct)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(r+ct,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r+ct > 0 \\ -(-r-ct)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(-r-ct,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r+ct < 0 \end{cases}$$

and

$$A_{\text{odd}}(r-ct) = \begin{cases} (r-ct)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(r-ct,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r-ct > 0 \\ -(-r+ct)\displaystyle\int_0^\pi \int_0^{2\pi} \alpha(-r+ct,\phi,\theta)\sin\theta\, d\phi\, d\theta & \text{if } r-ct < 0 \end{cases},$$

so for every region in the rt-quarter-plane, we have to test whether $r-ct$ and $r+ct$ are greater than or less than zero.

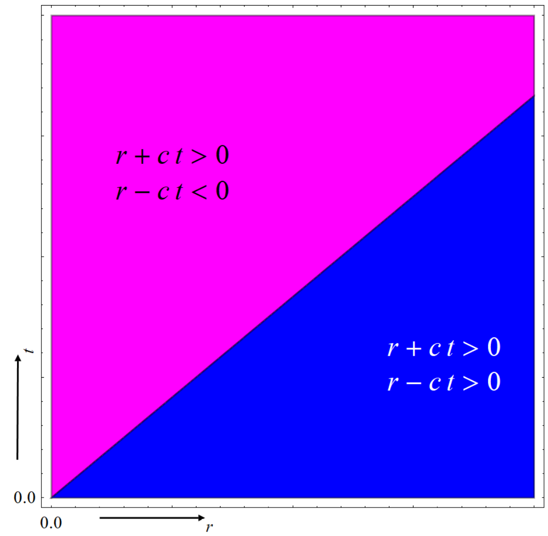


Figure 2: This figure illustrates the regions in the rt-quarter-plane that come about from using the odd extensions of A and B. The solution for w has to be determined in each one. The characteristic curve $r - ct = 0$ is the line that separates the regions.

The Magenta Region

In the magenta region $r + ct > 0$ and $r - ct < 0$, so the solution for w is

$$\begin{aligned} w(r,t) &= \frac{1}{2}[A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} B_{\text{odd}}(s)\,ds \\ &= \frac{1}{2}\left[(r+ct)\int_0^\pi \int_0^{2\pi} \alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta - (-r+ct)\int_0^\pi\int_0^{2\pi}\alpha(-r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\ &\qquad + \frac{1}{2c}\left[\int_{r-ct}^0 -(-s)\int_0^\pi\int_0^{2\pi}\beta(-s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds + \int_0^{r+ct} s\int_0^\pi\int_0^{2\pi}\beta(s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds\right]. \end{aligned}$$

Substitute $p = -s$ in the third integral and substitute $p = s$ in the fourth integral.

$$\begin{aligned} &= \frac{1}{2}\left[\int_0^\pi\int_0^{2\pi}(r+ct)\alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta - \int_0^\pi\int_0^{2\pi}(-r+ct)\alpha(-r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\ &\qquad + \frac{1}{2c}\left[\int_{-r+ct}^0 p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp + \int_0^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] \\ &= \frac{1}{2}\int_0^\pi\int_0^{2\pi}[(r+ct)\alpha(r+ct,\phi,\theta) - (-r+ct)\alpha(-r+ct,\phi,\theta)]\sin\theta\,d\phi\,d\theta \\ &\qquad + \frac{1}{2c}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp \end{aligned}$$

The expression in square brackets can be written as the derivative of an integral by the Leibnitz rule.

$$\begin{aligned} &= \frac{1}{2}\int_0^\pi\int_0^{2\pi}\left[\frac{1}{c}\frac{\partial}{\partial t}\int_{-r+ct}^{r+ct} p\alpha(p,\phi,\theta)\,dp\right]\sin\theta\,d\phi\,d\theta + \frac{1}{2c}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp \\ &= \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp \end{aligned}$$

The Blue Region

In the blue region $r + ct > 0$ and $r - ct > 0$, so the solution for w is

$$\begin{aligned}
w(r,t) &= \frac{1}{2}[A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c}\int_{r-ct}^{r+ct} B_{\text{odd}}(s)\,ds \\
&= \frac{1}{2}\left[(r+ct)\int_0^\pi \int_0^{2\pi} \alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta + (r-ct)\int_0^\pi \int_0^{2\pi} \alpha(r-ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\
&\quad + \frac{1}{2c}\int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds \\
&= \frac{1}{2}\left[\int_0^\pi \int_0^{2\pi} (r+ct)\alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta + \int_0^\pi \int_0^{2\pi} (r-ct)\alpha(r-ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\
&\quad + \frac{1}{2c}\int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds \\
&= \frac{1}{2}\int_0^\pi \int_0^{2\pi} [(r+ct)\alpha(r+ct,\phi,\theta) + (r-ct)\alpha(r-ct,\phi,\theta)]\sin\theta\,d\phi\,d\theta \\
&\quad + \frac{1}{2c}\int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s,\phi,\theta)\sin\theta\,d\phi\,d\theta\,ds.
\end{aligned}$$

The expression in square brackets can be written as the derivative of an integral by the Leibnitz rule. Also, let $p = s$ in the second integral.

$$\begin{aligned}
&= \frac{1}{2}\int_0^\pi \int_0^{2\pi} \left[\frac{1}{c}\frac{\partial}{\partial t}\int_{r-ct}^{r+ct} p\alpha(p,\phi,\theta)\,dp\right]\sin\theta\,d\phi\,d\theta + \frac{1}{2c}\int_{r-ct}^{r+ct} p\int_0^\pi \int_0^{2\pi} \beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp \\
&= \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{r-ct}^{r+ct} p\int_0^\pi \int_0^{2\pi} \alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} p\int_0^\pi \int_0^{2\pi} \beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp
\end{aligned}$$

Consequently,

$$w(r,t) = \begin{cases} \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{-r+ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp & \text{if } r-ct<0 \\ \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\int_{r-ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp & \text{if } r-ct>0 \end{cases}.$$

Since $v = w/r$, we have

$$\int_0^\pi \int_0^{2\pi} u(r,\phi,\theta,t)\sin\theta\,d\phi\,d\theta = \frac{w(r,t)}{r}.$$

In order to calculate the value of u at the origin, take the limit of both sides as $r \to 0$.

$$\lim_{r\to 0}\int_0^\pi \int_0^{2\pi} u(r,\phi,\theta,t)\sin\theta\,d\phi\,d\theta = \lim_{r\to 0}\frac{w(r,t)}{r}$$

The value of u at $r=0$ is $u(x=0,y=0,z=0,t)$. It does not depend on ϕ and θ, so it can be pulled in front of the integral.

$$u(0,0,0,t)\int_0^\pi \int_0^{2\pi} \sin\theta\,d\phi\,d\theta = \lim_{r\to 0}\frac{w(r,t)-w(0,t)}{r-0}$$

Evaluate the integral on the left side. The limit on the right side is how the first derivative of w with respect to r at $r=0$ is defined. The formula for w in the magenta region applies for this value of r. Use the Leibnitz rule to differentiate the integrals with respect to r.

$$\begin{aligned} 4\pi u(0,0,0,t) &= \left.\frac{\partial w}{\partial r}\right|_{r=0} \\ &= \left\{\frac{\partial}{\partial t}\left[\frac{1}{2c}\frac{\partial}{\partial r}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\alpha(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right] + \frac{1}{2c}\frac{\partial}{\partial r}\int_{-r+ct}^{r+ct} p\int_0^\pi\int_0^{2\pi}\beta(p,\phi,\theta)\sin\theta\,d\phi\,d\theta\,dp\right\}\Bigg|_{r=0} \\ &= \left\{\frac{1}{2c}\frac{\partial}{\partial t}\left[(r+ct)\int_0^\pi\int_0^{2\pi}\alpha(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta - (-r+ct)(-1)\int_0^\pi\int_0^{2\pi}\alpha(-r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right]\right. \\ &\qquad \left. + \frac{1}{2c}\left[(r+ct)\int_0^\pi\int_0^{2\pi}\beta(r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta - (-r+ct)(-1)\int_0^\pi\int_0^{2\pi}\beta(-r+ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right]\right\}\Bigg|_{r=0} \\ &= \frac{1}{2c}\frac{\partial}{\partial t}\left[ct\int_0^\pi\int_0^{2\pi}\alpha(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta + ct\int_0^\pi\int_0^{2\pi}\alpha(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \\ &\qquad + \frac{1}{2c}\left[ct\int_0^\pi\int_0^{2\pi}\beta(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta + ct\int_0^\pi\int_0^{2\pi}\beta(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] \end{aligned}$$

As a result,

$$\begin{aligned}4\pi u(0,0,0,t) &= \frac{\partial}{\partial t}\left[t\int_0^\pi \int_0^{2\pi} \alpha(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta\right] + t\int_0^\pi \int_0^{2\pi} \beta(ct,\phi,\theta)\sin\theta\,d\phi\,d\theta \\ &= \frac{\partial}{\partial t}\left[\frac{1}{c^2t}\int_0^\pi \int_0^{2\pi} \alpha(ct,\phi,\theta)(c^2t^2\sin\theta\,d\phi\,d\theta)\right] + \frac{1}{c^2t}\int_0^\pi \int_0^{2\pi} \beta(ct,\phi,\theta)(c^2t^2\sin\theta\,d\phi\,d\theta).\end{aligned}$$

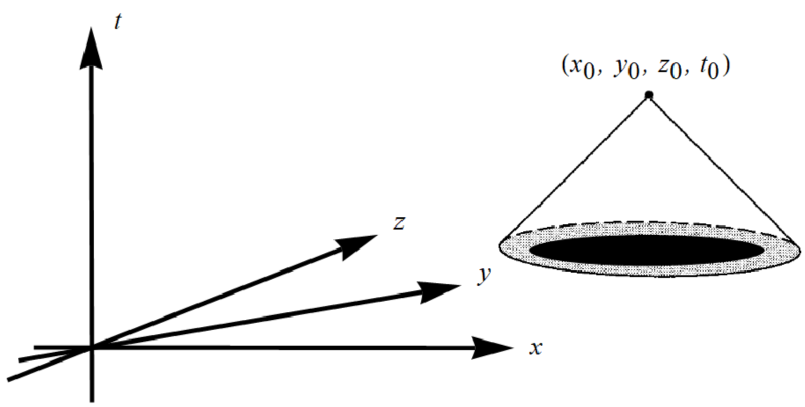
Divide both sides by 4π.

$$u(0,0,0,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2t}\int_0^\pi \int_0^{2\pi} \alpha(ct,\phi,\theta)(c^2t^2\sin\theta\,d\phi\,d\theta)\right] + \frac{1}{4\pi c^2t}\int_0^\pi \int_0^{2\pi} \beta(ct,\phi,\theta)(c^2t^2\sin\theta\,d\phi\,d\theta)$$

At a particular time $t = t_0$ these double integrals are surface integrals over a sphere of radius ct_0 centered at $(0,0,0)$.

$$u(0,0,0,t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2t_0}\iint\limits_{x^2+y^2+z^2=c^2t_0^2} \alpha(x,y,z)\,dS\right] + \frac{1}{4\pi c^2t_0}\iint\limits_{x^2+y^2+z^2=c^2t_0^2} \beta(x,y,z)\,dS$$

This is the solution of the wave equation at the origin of the xyz-plane. Now we aim to find the solution at a particular point in space-time (x_0, y_0, z_0, t_0).



The wave equation is invariant to translations in space, so if $u(x, y, z, t)$ is a solution to the wave equation

$$\begin{aligned}&u_{tt} = c^2\nabla^2 u, \quad -\infty < x,y,z < \infty,\ t > 0 \\ &u(x,y,z,0) = \alpha(x,y,z) \\ &u_t(x,y,z,0) = \beta(x,y,z),\end{aligned}$$

then $u(x + x_0, y + y_0, z + z_0, t)$ is also a solution to the wave equation, albeit with different initial conditions.

$$\begin{aligned}&u_{tt} = c^2\nabla^2 u, \quad -\infty < x,y,z < \infty,\ t > 0 \\ &u(x+x_0, y+y_0, z+z_0, 0) = \alpha(x+x_0, y+y_0, z+z_0) \\ &u_t(x+x_0, y+y_0, z+z_0, 0) = \beta(x+x_0, y+y_0, z+z_0)\end{aligned}$$

Since the solution to the wave equation is unique, $u(x_0, y_0, z_0, t_0)$ has the same form as $u(0, 0, 0, t_0)$.

$$u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \iint\limits_{x^2+y^2+z^2=c^2t_0^2} \alpha(x + x_0, y + y_0, z + z_0)\, dS\right] + \frac{1}{4\pi c^2 t_0} \iint\limits_{x^2+y^2+z^2=c^2t_0^2} \beta(x + x_0, y + y_0, z + z_0)\, dS$$

Let $k = x + x_0$, $l = y + y_0$, and $m = z + z_0$. Then $x = k - x_0$, $y = l - y_0$, and $z = m - z_0$.

$$u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \iint\limits_{\substack{(k-x_0)^2+(l-y_0)^2\\+(m-z_0)^2=c^2t_0^2}} \alpha(k, l, m)\, dS\right] + \frac{1}{4\pi c^2 t_0} \iint\limits_{\substack{(k-x_0)^2+(l-y_0)^2\\+(m-z_0)^2=c^2t_0^2}} \beta(k, l, m)\, dS$$

k, l, and m are dummy integration variables, so they can be replaced with x, y, and z.

$$u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0}\left[\frac{1}{4\pi c^2 t_0} \iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=c^2t_0^2}} \alpha(x, y, z)\, dS\right] + \frac{1}{4\pi c^2 t_0} \iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2\\+(z-z_0)^2=c^2t_0^2}} \beta(x, y, z)\, dS$$

Finally, switch the roles of x, y, z, and t with those of x_0, y_0, z_0, and t_0, respectively, to obtain the legendary solution to the initial value problem (discovered by Kirchhoff & Poisson).

$$\boxed{u(x, y, z, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \alpha(x_0, y_0, z_0)\, dS_0\right] + \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} \beta(x_0, y_0, z_0)\, dS_0}$$

These double integrals are surface integrals over a sphere of radius ct centered at (x, y, z). In this exercise $\alpha = 0$ and $\beta = x^2 + y^2 + z^2$, so the formula simplifies to

$$u(x, y, z, t) = \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\+(z_0-z)^2=c^2t^2}} (x_0^2 + y_0^2 + z_0^2)\, dS_0.$$

Use spherical coordinates (r_0, ϕ_0, θ_0) to evaluate the integral.

$$\begin{aligned}
x_0 - x &= ct\sin\theta_0\cos\phi_0 & &\rightarrow & x_0 &= x + ct\sin\theta_0\cos\phi_0 & &\rightarrow & x_0^2 &= x^2 + 2ctx\sin\theta_0\cos\phi_0 + c^2t^2\sin^2\theta_0\cos^2\phi_0 \\
y_0 - y &= ct\sin\theta_0\sin\phi_0 & &\rightarrow & y_0 &= y + ct\sin\theta_0\sin\phi_0 & &\rightarrow & y_0^2 &= y^2 + 2cty\sin\theta_0\sin\phi_0 + c^2t^2\sin^2\theta_0\sin^2\phi_0 \\
z_0 - z &= ct\cos\theta_0 & &\rightarrow & z_0 &= z + ct\cos\theta_0 & &\rightarrow & z_0^2 &= z^2 + 2ctz\cos\theta_0 + c^2t^2\cos^2\theta_0
\end{aligned}$$

So then $x_0^2 + y_0^2 + z_0^2 = x^2 + y^2 + z^2 + 2ct(x\sin\theta_0\cos\phi_0 + y\sin\theta_0\sin\phi_0 + z\cos\theta_0) + c^2t^2$.

As a result,

$$
\begin{aligned}
u(x,y,z,t) &= \frac{1}{4\pi c^2 t}\int_0^\pi \int_0^{2\pi} [x^2+y^2+z^2+c^2t^2+2ct(x\sin\theta_0\cos\phi_0+y\sin\theta_0\sin\phi_0+z\cos\theta_0)](c^2t^2\sin\theta_0\,d\phi_0\,d\theta_0) \\
&= \frac{t}{4\pi}\int_0^\pi \int_0^{2\pi} [(x^2+y^2+z^2+c^2t^2)\sin\theta_0+2ct(x\sin^2\theta_0\cos\phi_0+y\sin^2\theta_0\sin\phi_0+z\sin\theta_0\cos\theta_0)]\,d\phi_0\,d\theta_0 \\
&= \frac{t}{4\pi}\left[(x^2+y^2+z^2+c^2t^2)\int_0^\pi \int_0^{2\pi} \sin\theta_0\,d\phi_0\,d\theta_0\right. \\
&\qquad \left.+2ct\left(x\int_0^\pi \int_0^{2\pi} \sin^2\theta_0\cos\phi_0\,d\phi_0\,d\theta_0+y\int_0^\pi \int_0^{2\pi} \sin^2\theta_0\sin\phi_0\,d\phi_0\,d\theta_0+z\int_0^\pi \int_0^{2\pi} \sin\theta_0\cos\theta_0\,d\phi_0\,d\theta_0\right)\right] \\
&= \frac{t}{4\pi}\left\{(x^2+y^2+z^2+c^2t^2)\left(\int_0^\pi \sin\theta_0\,d\theta_0\right)\left(\int_0^{2\pi} d\phi_0\right)\right. \\
&\qquad \left.+2ct\left[x\left(\int_0^\pi \sin^2\theta_0\,d\theta_0\right)\underbrace{\left(\int_0^{2\pi}\cos\phi_0\,d\phi_0\right)}_{=0}+y\left(\int_0^\pi \sin^2\theta_0\,d\theta_0\right)\underbrace{\left(\int_0^{2\pi}\sin\phi_0\,d\phi_0\right)}_{=0}+z\underbrace{\left(\int_0^\pi \sin\theta_0\cos\theta_0\,d\theta_0\right)}_{=0}\left(\int_0^{2\pi} d\phi_0\right)\right]\right\} \\
&= \frac{t}{4\pi}[(x^2+y^2+z^2+c^2t^2)(2)(2\pi)].
\end{aligned}
$$

Therefore, the solution to the initial value problem with $\alpha = 0$ and $\beta = x^2+y^2+z^2$ is

$$
u(x,y,z,t) = t(x^2+y^2+z^2+c^2t^2).
$$