Exercise 8

Carry out the derivation of the second term in (3).

Solution

Equation (3) is the solution to the three-dimensional wave equation in space subject to two initial conditions.

\[ u_{tt} = c^2 \nabla^2 u, \quad -\infty < x, y, z < \infty, \quad t > 0 \]
\[ u(x, y, z, 0) = \alpha(x, y, z) \]
\[ u_t(x, y, z, 0) = \beta(x, y, z) \]

Integrate both sides of the wave equation over the black hyperdisk shown in the figure below.

Figure 1: The black hyperdisk lies in the \( xyz \)-plane, has center \((0, 0, 0)\) and radius \( r \), and represents a solid ball in \( xyz \)-space. Note that the shaded hyperdisk it lies within has radius \( ct_0 \), where \( t_0 \) is a particular time we want to evaluate \( u \) at.

\[
\iiint_V u_{tt} \, dV = \iiint_V c^2 \nabla^2 u \, dV
\]
\[
\iiint_V u_{tt} \, dV = c^2 \iiint_V \nabla \cdot \nabla u \, dV
\]

Apply the divergence theorem to the volume integral on the right side to turn it into a surface integral over the solid ball’s boundary.

\[
\iiint_V u_{tt} \, dV = c^2 \oint_S \nabla u \cdot \hat{n} \, dS
\]

The unit vector normal to the boundary is the radial unit vector: \( \hat{n} = \hat{r} \). \( \nabla u \cdot \hat{r} \) can be interpreted as the directional derivative in the radial direction, that is, \( \partial u / \partial r \).

\[
\iiint_V \frac{\partial^2 u}{\partial t^2} \, dV = c^2 \oint_S \frac{\partial u}{\partial r} \, dS
\]
Write out the volume and surface integrals explicitly by using spherical coordinates \((r, \phi, \theta)\). Here \(\theta\) denotes the angle from the polar axis.

\[
\int_0^\pi \int_0^{2\pi} \int_0^r \frac{\partial^2 u}{\partial \rho^2} (\rho^2 \sin \theta \, d\rho \, d\phi \, d\theta) = c^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r} (r^2 \sin \theta \, d\phi \, d\theta)
\]

\[
\int_0^r \rho^2 \int_0^\pi \int_0^{2\pi} \sin \theta \, d\phi \, d\theta \, d\rho = c^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r} \sin \theta \, d\phi \, d\theta
\]

\[
\int_0^r \rho^2 \frac{\partial^2 u}{\partial r^2} \left[ \int_0^\pi \int_0^{2\pi} u(\rho, \phi, \theta, t) \sin \theta \, d\phi \, d\theta \right] \, d\rho = c^2 r^2 \frac{\partial}{\partial r} \left[ \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta \right]
\]

Let

\[
v(r, t) = \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta
\]

so that the previous equation becomes

\[
\int_0^r \rho^2 \frac{\partial^2 v}{\partial t^2} \, d\rho = c^2 r^2 \frac{\partial v}{\partial r}.
\]

Differentiate both sides with respect to \(r\) to eliminate the integral on the left side.

\[
r^2 \frac{\partial^2 v}{\partial t^2} = c^2 \left( 2r \frac{\partial v}{\partial r} + r^2 \frac{\partial^2 v}{\partial r^2} \right)
\]

Divide both sides by \(r\).

\[
r \frac{\partial^2 v}{\partial t^2} = c^2 \left( 2 \frac{\partial v}{\partial r} + r \frac{\partial^2 v}{\partial r^2} \right)
\]

Now make the change of variables \(w(r, t) = rv(r, t)\). Write the new derivatives in terms of the old ones by differentiating this substitution.

\[
\frac{\partial^2 w}{\partial t^2} = \frac{r^2}{r} \frac{\partial^2 v}{\partial r^2} - \frac{2r}{r} \frac{\partial v}{\partial r} + \frac{\partial v}{\partial r}
\]

The transformed PDE is the wave equation on a semi-infinite interval

\[
\frac{\partial^2 w}{\partial t^2} = c^2 \frac{\partial^2 w}{\partial r^2}, \quad 0 < r < \infty, \quad t > 0
\]

subject to the Dirichlet boundary condition \(w(0, t) = 0\) and the initial conditions,

\[
w(r, 0) = r \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, 0) \sin \theta \, d\phi \, d\theta \quad \text{and} \quad w_t(r, 0) = r \int_0^\pi \int_0^{2\pi} u_t(r, \phi, \theta, 0) \sin \theta \, d\phi \, d\theta
\]

\[
= r \int_0^\pi \int_0^{2\pi} \alpha(r, \phi, \theta) \sin \theta \, d\phi \, d\theta \quad \text{and} \quad = r \int_0^\pi \int_0^{2\pi} \beta(r, \phi, \theta) \sin \theta \, d\phi \, d\theta.
\]
The method of reflection can be applied to solve this one-dimensional wave equation. Consider the corresponding problem over the whole line,

\[ W_{tt} = c^2 W_{rr}, \quad -\infty < r < \infty, \ t > 0 \]

\[ W(r,0) = A_{\text{odd}}(r), \quad W_t(r,0) = B_{\text{odd}}(r), \]

where the odd extensions of the initial conditions for \( w \), \( A_{\text{odd}}(r) \) and \( B_{\text{odd}}(r) \), are used in order to satisfy the Dirichlet boundary condition.

\[
A_{\text{odd}}(r) = \begin{cases} 
  r \int_0^\pi \int_0^{2\pi} \alpha(r,\phi,\theta) \sin \theta \ d\phi \ d\theta & \text{if } r > 0 \\
  -(-r) \int_0^\pi \int_0^{2\pi} \alpha(-r,\phi,\theta) \sin \theta \ d\phi \ d\theta & \text{if } r < 0
\end{cases}
\]

\[
B_{\text{odd}}(r) = \begin{cases} 
  r \int_0^\pi \int_0^{2\pi} \beta(r,\phi,\theta) \sin \theta \ d\phi \ d\theta & \text{if } r > 0 \\
  -(-r) \int_0^\pi \int_0^{2\pi} \beta(-r,\phi,\theta) \sin \theta \ d\phi \ d\theta & \text{if } r < 0
\end{cases}
\]

The solution for \( W \) is given by d’Alembert’s formula in section 2.1 on page 36.

\[
W(r,t) = \frac{1}{2} [A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} B_{\text{odd}}(s) \ ds
\]

The solution for \( w \) is then just the restriction of \( W \) to \( r > 0 \).

\[
w(r,t) = \frac{1}{2} [A_{\text{odd}}(r+ct) + A_{\text{odd}}(r-ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} B_{\text{odd}}(s) \ ds, \quad r > 0
\]

Our task now is to write this formula in terms of the given functions, \( \alpha \) and \( \beta \). Note that

\[
A_{\text{odd}}(r+ct) = \begin{cases} 
  (r+ct) \int_0^\pi \int_0^{2\pi} \alpha(r+ct,\phi,\theta) \sin \theta \ d\phi \ d\theta & \text{if } r+ct > 0 \\
  -(-r+ct) \int_0^\pi \int_0^{2\pi} \alpha(-r+ct,\phi,\theta) \sin \theta \ d\phi \ d\theta & \text{if } r+ct < 0
\end{cases}
\]

and

\[
A_{\text{odd}}(r-ct) = \begin{cases} 
  (r-ct) \int_0^\pi \int_0^{2\pi} \alpha(r-ct,\phi,\theta) \sin \theta \ d\phi \ d\theta & \text{if } r-ct > 0 \\
  -(-r+ct) \int_0^\pi \int_0^{2\pi} \alpha(-r+ct,\phi,\theta) \sin \theta \ d\phi \ d\theta & \text{if } r-ct < 0
\end{cases}
\]

so for every region in the \( rt \)-quarter-plane, we have to test whether \( r-ct \) and \( r+ct \) are greater than or less than zero.
Figure 2: This figure illustrates the regions in the $rt$-quarter-plane that come about from using the odd extensions of $A$ and $B$. The solution for $w$ has to be determined in each one. The characteristic curve $r - ct = 0$ is the line that separates the regions.

**The Magenta Region**

In the magenta region $r + ct > 0$ and $r - ct < 0$, so the solution for $w$ is

$$w(r, t) = \frac{1}{2} [A_{\text{odd}}(r + ct) + A_{\text{odd}}(r - ct)] + \frac{1}{2c} \int_{r - ct}^{r + ct} B_{\text{odd}}(s) \, ds$$

$$= \frac{1}{2} \left[ (r + ct) \int_0^\pi \int_0^{2\pi} \alpha(r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta - (-r + ct) \int_0^\pi \int_0^{2\pi} \alpha(-r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right]$$

$$+ \frac{1}{2c} \left[ \int_{r - ct}^0 (-s) \int_0^\pi \int_0^{2\pi} \beta(-s, \phi, \theta) \sin \theta \, d\phi \, d\theta \, ds + \int_0^{r + ct} s \int_0^\pi \int_0^{2\pi} \beta(s, \phi, \theta) \sin \theta \, d\phi \, d\theta \, ds \right].$$

Substitute $p = -s$ in the third integral and substitute $p = s$ in the fourth integral.

$$= \frac{1}{2} \left[ \int_0^\pi \int_0^{2\pi} (r + ct) \alpha(r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta - \int_0^\pi \int_0^{2\pi} (-r + ct) \alpha(-r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right]$$

$$+ \frac{1}{2c} \left[ \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \right]$$

$$= \frac{1}{2} \int_0^\pi \int_0^{2\pi} [(r + ct) \alpha(r + ct, \phi, \theta) - (-r + ct) \alpha(-r + ct, \phi, \theta)] \sin \theta \, d\phi \, d\theta$$

$$+ \frac{1}{2c} \int_{r - ct}^{r + ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp$$

The expression in square brackets can be written as the derivative of an integral by the Leibnitz rule.

$$= \frac{1}{2} \int_0^\pi \int_0^{2\pi} \left[ \frac{1}{c} \frac{\partial}{\partial t} \int_{r - ct}^{r + ct} p \alpha(p, \phi, \theta) \, dp \right] \sin \theta \, d\phi \, d\theta + \frac{1}{2c} \int_{r - ct}^{r + ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp$$

$$= \frac{\partial}{\partial t} \left[ \frac{1}{2c} \int_{r - ct}^{r + ct} p \int_0^\pi \int_0^{2\pi} \alpha(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp \right] + \frac{1}{2c} \int_{r - ct}^{r + ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp$$
The Blue Region

In the blue region \( r + ct > 0 \) and \( r - ct > 0 \), so the solution for \( w \) is

\[
w(r,t) = \frac{1}{2} [A_{\text{odd}}(r + ct) + A_{\text{odd}}(r - ct)] + \frac{1}{2c} \int_{r-ct}^{r+ct} B_{\text{odd}}(s) \, ds
\]

\[
= \frac{1}{2} \left[ (r + ct) \int_0^\pi \int_0^{2\pi} \alpha(r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta + (r - ct) \int_0^\pi \int_0^{2\pi} \alpha(r - ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right] \\
+ \frac{1}{2c} \int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s, \phi, \theta) \sin \theta \, d\phi \, d\theta \, ds
\]

\[
= \frac{1}{2} \left[ \int_0^\pi \int_0^{2\pi} (r + ct) \alpha(r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta + \int_0^\pi \int_0^{2\pi} (r - ct) \alpha(r - ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right] \\
+ \frac{1}{2c} \int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s, \phi, \theta) \sin \theta \, d\phi \, d\theta \, ds
\]

\[
= \frac{1}{2} \int_0^\pi \int_0^{2\pi} [(r + ct) \alpha(r + ct, \phi, \theta) + (r - ct) \alpha(r - ct, \phi, \theta)] \sin \theta \, d\phi \, d\theta \\
+ \frac{1}{2c} \int_{r-ct}^{r+ct} s \int_0^\pi \int_0^{2\pi} \beta(s, \phi, \theta) \sin \theta \, d\phi \, d\theta \, ds.
\]

The expression in square brackets can be written as the derivative of an integral by the Leibnitz rule. Also, let \( p = s \) in the second integral.

\[
= \frac{1}{2} \int_0^\pi \int_0^{2\pi} [\frac{1}{c} \frac{\partial}{\partial t} \int_{r-ct}^{r+ct} p \alpha(p, \phi, \theta) \, dp] \sin \theta \, d\phi \, d\theta + \frac{1}{2c} \int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp
\]

\[
= \frac{\partial}{\partial t} \left[ \frac{1}{2c} \int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \alpha(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp \right] + \frac{1}{2c} \int_{r-ct}^{r+ct} p \int_0^\pi \int_0^{2\pi} \beta(p, \phi, \theta) \sin \theta \, d\phi \, d\theta \, dp
\]
Consequently,

\[
\begin{align*}
\frac{\partial}{\partial t} \left[ \frac{1}{2c} \int_{r - ct}^{r + ct} r \int_0^\pi \alpha(p, \phi, \theta) \sin \theta \, d\phi \, dp \right] + \frac{1}{2c} \int_{r - ct}^{r + ct} \beta(p, \phi, \theta) \sin \theta \, d\phi \, dp & \quad \text{if } r - ct < 0 \\
\frac{\partial}{\partial t} \left[ \frac{1}{2c} \int_{r - ct}^{r + ct} r \int_0^\pi \alpha(p, \phi, \theta) \sin \theta \, d\phi \, dp \right] + \frac{1}{2c} \int_{r - ct}^{r + ct} \beta(p, \phi, \theta) \sin \theta \, d\phi \, dp & \quad \text{if } r - ct > 0
\end{align*}
\]

Since \( v = w/r \), we have

\[
\int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta = \frac{w(r, t)}{r}.
\]

In order to calculate the value of \( u \) at the origin, take the limit of both sides as \( r \to 0 \).

\[
\lim_{r \to 0} \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta = \lim_{r \to 0} \frac{w(r, t)}{r}
\]

The value of \( u \) at \( r = 0 \) is \( u(x = 0, y = 0, z = 0, t) \). It does not depend on \( \phi \) and \( \theta \), so it can be pulled in front of the integral.

\[
u(0, 0, 0, t) \int_0^\pi \int_0^{2\pi} \sin \theta \, d\phi \, d\theta = \lim_{r \to 0} \frac{w(r, t) - w(0, t)}{r - 0}
\]

Evaluate the integral on the left side. The limit on the right side is how the first derivative of \( w \) with respect to \( r \) at \( r = 0 \) is defined.

The formula for \( w \) in the magenta region applies for this value of \( r \). Use the Leibnitz rule to differentiate the integrals with respect to \( r \).

\[
4\pi u(0, 0, 0, t) = \frac{\partial w}{\partial r} \bigg|_{r=0} = \left\{ \frac{\partial}{\partial t} \left[ \frac{1}{2c} \int_{r - ct}^{r + ct} r \int_0^\pi \alpha(p, \phi, \theta) \sin \theta \, d\phi \, dp \right] + \frac{1}{2c} \int_{r - ct}^{r + ct} \beta(p, \phi, \theta) \sin \theta \, d\phi \, dp \right\} \bigg|_{r=0}
\]

\[
= \frac{1}{2c} \frac{\partial}{\partial t} \left[ (r + ct) \int_0^\pi \int_0^{2\pi} \alpha(r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta - (r - ct)(-1) \int_0^\pi \int_0^{2\pi} \alpha(-r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right]
\]

\[
+ \frac{1}{2c} \left[ (r + ct) \int_0^\pi \int_0^{2\pi} \beta(r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta - (r - ct)(-1) \int_0^\pi \int_0^{2\pi} \beta(-r + ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right] \bigg|_{r=0}
\]

\[
= \frac{1}{2c} \frac{\partial}{\partial t} \left[ ct \int_0^\pi \int_0^{2\pi} \alpha(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta + ct \int_0^\pi \int_0^{2\pi} \alpha(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right]
\]

\[
+ \frac{1}{2c} \left[ ct \int_0^\pi \int_0^{2\pi} \beta(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta + ct \int_0^\pi \int_0^{2\pi} \beta(ct, \phi, \theta) \sin \theta \, d\phi \, d\theta \right]
\]
As a result,

$$4\pi u(0,0,0,t) = \frac{\partial}{\partial t} \left[ t \int_0^\pi \int_0^{2\pi} \alpha(ct,\phi,\theta) \sin \theta \, d\phi \, d\theta \right] + t \int_0^\pi \int_0^{2\pi} \beta(ct,\phi,\theta) \sin \theta \, d\phi \, d\theta$$

$$= \frac{\partial}{\partial t} \left[ \frac{1}{c^2t} \int_0^\pi \int_0^{2\pi} \alpha(ct,\phi,\theta) (c^2t^2 \sin \theta) \, d\phi \, d\theta \right] + \frac{1}{c^2t} \int_0^\pi \int_0^{2\pi} \beta(ct,\phi,\theta) (c^2t^2 \sin \theta) \, d\phi \, d\theta.$$ 

Divide both sides by $4\pi$.

$$u(0,0,0,t) = \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c^2t} \int_0^\pi \int_0^{2\pi} \alpha(ct,\phi,\theta) (c^2t^2 \sin \theta) \, d\phi \, d\theta \right] + \frac{1}{4\pi c^2t} \int_0^\pi \int_0^{2\pi} \beta(ct,\phi,\theta) (c^2t^2 \sin \theta) \, d\phi \, d\theta$$

At a particular time $t = t_0$ these double integrals are surface integrals over a sphere of radius $ct_0$ centered at $(0,0,0)$.

$$u(0,0,0,t_0) = \frac{\partial}{\partial t_0} \left[ \frac{1}{4\pi c^2t_0} \iint_{x^2+y^2+z^2=c^2t_0^2} \alpha(x,y,z) \, dS \right] + \frac{1}{4\pi c^2t_0} \iint_{x^2+y^2+z^2=c^2t_0^2} \beta(x,y,z) \, dS$$

This is the solution of the wave equation at the origin of the $xyz$-plane. Now we aim to find the solution at a particular point in space-time $(x_0,y_0,z_0,t_0)$.

The wave equation is invariant to translations in space, so if $u(x,y,z,t)$ is a solution to the wave equation

$$u_{tt} = c^2 \nabla^2 u, \quad -\infty < x, y, z < \infty, \quad t > 0$$

$$u(x,y,z,0) = \alpha(x,y,z)$$

$$u_t(x,y,z,0) = \beta(x,y,z),$$

then $u(x+x_0,y+y_0,z+z_0,t)$ is also a solution to the wave equation, albeit with different initial conditions.

$$u_{tt} = c^2 \nabla^2 u, \quad -\infty < x, y, z < \infty, \quad t > 0$$

$$u(x+x_0,y+y_0,z+z_0,0) = \alpha(x+x_0,y+y_0,z+z_0)$$

$$u_t(x+x_0,y+y_0,z+z_0,0) = \beta(x+x_0,y+y_0,z+z_0)$$
Since the solution to the wave equation is unique, \( u(x_0, y_0, z_0, t_0) \) has the same form as \( u(0, 0, 0, t_0) \).

\[
u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0} \left[ \frac{1}{4\pi c^2 t_0} \int \int \alpha(x + x_0, y + y_0, z + z_0) \, dS \right] + \frac{1}{4\pi c^2 t_0} \int \int \beta(x + x_0, y + y_0, z + z_0) \, dS
\]

Let \( k = x + x_0, l = y + y_0, \) and \( m = z + z_0 \). Then \( x = k - x_0, y = l - y_0, \) and \( z = m - z_0 \).

\[
u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0} \left[ \frac{1}{4\pi c^2 t_0} \int \int \alpha(k, l, m) \, dS \right] + \frac{1}{4\pi c^2 t_0} \int \int \beta(k, l, m) \, dS
\]

\( k, l, \) and \( m \) are dummy integration variables, so they can be replaced with \( x, y, \) and \( z \).

\[
u(x_0, y_0, z_0, t_0) = \frac{\partial}{\partial t_0} \left[ \frac{1}{4\pi c^2 t_0} \int \int \alpha(x, y, z) \, dS \right] + \frac{1}{4\pi c^2 t_0} \int \int \beta(x, y, z) \, dS
\]

Finally, switch the roles of \( x, y, z, \) and \( t \) with those of \( x_0, y_0, z_0, \) and \( t_0 \), respectively, to obtain the legendary solution to the initial value problem (discovered by Kirchhoff & Poisson).

\[
u(x, y, z, t) = \frac{\partial}{\partial t} \left[ \frac{1}{4\pi c^2 t} \int \int \alpha(x_0, y_0, z_0) \, dS_0 \right] + \frac{1}{4\pi c^2 t} \int \int \beta(x_0, y_0, z_0) \, dS_0
\]

These double integrals are surface integrals over a sphere of radius \( ct \) centered at \( (x, y, z) \).