Exercise 10

Derive the mean value property of harmonic functions $u(x, y, z)$ by the following method. A harmonic function is a wave that happens not to depend on time, so that its mean value $\overline{u}(r, t) = \overline{u}(r)$ satisfies (5). Deduce that $\overline{u}(r) = u(0)$.

Solution

$\overline{u}(r, t)$ is defined to be the average of $u$ over a spherical surface of radius $r$. Spherical coordinates $(r, \phi, \theta)$ will be used to write the forthcoming integrals explicitly, where $\theta$ represents the angle from the polar axis.

$$\overline{u}(r, t) = \frac{\iiint u \, dS}{\iiint dS} = \frac{\int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) (r^2 \sin \theta \, d\phi \, d\theta)}{\int_0^\pi \int_0^{2\pi} (r^2 \sin \theta \, d\phi \, d\theta)} = \frac{\int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta}{\int_0^\pi \sin \theta \, d\theta \left( \int_0^{2\pi} d\phi \right)}$$

Evaluating the integrals in the denominator, we have

$$\overline{u}(r, t) = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta. \quad (1)$$

The three-dimensional wave equation in space is

$$u_{tt} = c^2 \nabla^2 u, \quad -\infty < x, y, z < \infty, \quad t > 0.$$

Integrate both sides over the volume enclosed by that sphere of radius $r$.

$$\iiint_V u_{tt} \, dV = \iiint_V c^2 \nabla^2 u \, dV$$

$$\iiint_V u_{tt} \, dV = c^2 \iiint_V \nabla \cdot \nabla u \, dV$$

Apply the divergence theorem to the volume integral on the right side to turn it into a surface integral over the sphere’s boundary.

$$\iiint_V u_{tt} \, dV = c^2 \int_S \nabla u \cdot \hat{n} \, dS$$

The unit vector normal to the boundary is the radial unit vector: $\hat{n} = \hat{r}$. $\nabla u \cdot \hat{r}$ can be interpreted as the directional derivative in the radial direction, that is, $\partial u/\partial r$.

$$\iiint_V \frac{\partial^2 u}{\partial t^2} \, dV = c^2 \oint_S \frac{\partial u}{\partial r} \, dS$$

$$\int_0^\pi \int_0^{2\pi} \int_0^r \frac{\partial^2 u}{\partial t^2} (r^2 \sin \theta \, d\rho \, d\phi \, d\theta) = c^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r} (r^2 \sin \theta \, d\phi \, d\theta)$$

$$\int_0^r \rho^2 \int_0^\pi \int_0^{2\pi} \frac{\partial^2 u}{\partial t^2} \sin \theta \, d\phi \, d\theta \, d\rho = c^2 r^2 \int_0^\pi \int_0^{2\pi} \frac{\partial u}{\partial r} \sin \theta \, d\phi \, d\theta$$

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\[ \int_0^r \rho^2 \frac{\partial^2}{\partial t^2} \left[ \int_0^\pi \int_0^{2\pi} u(\rho, \phi, \theta, t) \sin \theta \, d\phi \, d\theta \right] \, d\rho = c^2 r^2 \frac{\partial}{\partial r} \left[ \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta \right] \]

Divide both sides by \(4\pi\).

\[ \int_0^r \rho^2 \frac{\partial^2}{\partial t^2} \left[ \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} u(\rho, \phi, \theta, t) \sin \theta \, d\phi \, d\theta \right] \, d\rho = c^2 r^2 \frac{\partial}{\partial r} \left[ \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} u(r, \phi, \theta, t) \sin \theta \, d\phi \, d\theta \right] \]

Substitute equation (1) here.

\[ \int_0^r \rho^2 \frac{\partial^2 u}{\partial t^2} \, d\rho = c^2 r^2 \frac{\partial u}{\partial r} \]

Suppose now that \(u\) is a harmonic function, a wave that does not depend on time. Then \(\bar{u} = \bar{u}(r)\), and the left side is equal to zero.

\[ 0 = c^2 r^2 \frac{d\bar{u}}{dr} \]

Divide both sides by \(c^2 r^2\).

\[ \frac{d\bar{u}}{dr} = 0 \]

Integrate both sides with respect to \(r\).

\[ \bar{u}(r) = C_1 \]

We conclude that the average of \(u\) (a harmonic function) over a spherical surface is the same at every radius, including \(r = 0\).

\[ \bar{u}(r) = \bar{u}(0) \]

\[ = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} u(0, \phi, \theta) \sin \theta \, d\phi \, d\theta \]

Note that \(u\) evaluated at \(r = 0\) is \(u(x = 0, y = 0, z = 0)\), or \(u(0)\), and does not depend on \(\phi\) or \(\theta\).

\[ = \frac{1}{4\pi} \int_0^\pi \int_0^{2\pi} u(0) \sin \theta \, d\phi \, d\theta \]

\[ = \frac{u(0)}{4\pi} \int_0^\pi \int_0^{2\pi} \sin \theta \, d\phi \, d\theta \]

\[ = \frac{u(0)}{4\pi} \left( \int_0^\pi \sin \theta \, d\theta \right) \left( \int_0^{2\pi} d\phi \right) \]

\[ = \frac{u(0)}{4\pi} (2)(2\pi) \]

Therefore,

\[ \bar{u}(r) = u(0). \]