Exercise 13

Solve the wave equation in the half-space $\{(x, y, z, t) : \ z > 0\}$ with the Neumann condition $\partial u/\partial z = 0$ on $z = 0$, and with initial data $\phi(x, y, z) \equiv 0$ and general $\psi(x, y, z)$. (*Hint:* See (3) and use the method of reflection.)

Solution

Here we have to solve

$$\begin{gathered} u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y < \infty, \ 0 < z < \infty, \ t > 0 \\ u(x, y, z, 0) = 0 \qquad u_t(x, y, z, 0) = \psi(x, y, z) \\ u_z(x, y, 0, t) = 0. \end{gathered}$$

Since z is over a semi-infinite interval, we can use the method of reflection to find the solution. Consider the same problem over an infinite interval, using the even extension of ϕ in order to satisfy the Neumann boundary condition at $z = 0$:

$$\begin{gathered} w_{tt} = c^2\nabla^2 w, \quad -\infty < x, y, z < \infty, \ t > 0 \\ w(x, y, z, 0) = 0 \qquad w_t(x, y, z, 0) = \psi_{\text{even}}(x, y, z), \end{gathered}$$

where

$$\psi_{\text{even}}(x, y, z) = \begin{cases} \psi(x, y, z) & \text{if } z > 0 \\ \psi(x, y, -z) & \text{if } z < 0 \end{cases}.$$

The solution for w is given by the formula of Kirchhoff and Poisson.

$$\begin{aligned} w(x, y, z, t) &= \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} (0)\, dS_0\right] + \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \psi_{\text{even}}(x_0, y_0, z_0)\, dS_0 \\ &= \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \psi_{\text{even}}(x_0, y_0, z_0)\, dS_0 \end{aligned}$$

The solution for u is then just the restriction of w to $z > 0$.

$$u(x, y, z, t) = \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} \psi_{\text{even}}(x_0, y_0, z_0)\, dS_0, \quad z > 0$$

Our task now is to write this formula in terms of the given function $\psi(x, y, z)$. Split up the surface integral into two parts, one over the part of the sphere that lies in the upper half of $x_0y_0z_0$-space and one over the part of the sphere that lies in the lower half of $x_0y_0z_0$-space.

$$u(x, y, z, t) = \frac{1}{4\pi c^2 t}\left[\iint\limits_{S_1} \psi(x_0, y_0, z_0)\, dS_0 + \iint\limits_{S_2} \psi(x_0, y_0, -z_0)\, dS_0\right], \quad z > 0$$

Specifically, S_1 and S_2 are defined as follows.

$$\begin{gathered} S_1 = \{(x_0, y_0, z_0) \mid (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = c^2t^2, \ z_0 > 0\} \\ S_2 = \{(x_0, y_0, z_0) \mid (x_0 - x)^2 + (y_0 - y)^2 + (z_0 - z)^2 = c^2t^2, \ z_0 < 0\} \end{gathered}$$

Replace z_0 with $-z_0$ in the second surface integral.

$$u(x,y,z,t)=\frac{1}{4\pi c^2 t}\left[\iint_{S_1}\psi(x_0,y_0,z_0)\,dS_0+\iint_{S_3}\psi(x_0,y_0,z_0)\,dS_0\right],\quad z>0$$

S_3 is the sphere defined as

$$\begin{aligned}S_3&=\{(x_0,y_0,z_0)\mid(x_0-x)^2+(y_0-y)^2+(-z_0-z)^2=c^2t^2,\ -z_0<0\}\\&=\{(x_0,y_0,z_0)\mid(x_0-x)^2+(y_0-y)^2+(z_0+z)^2=c^2t^2,\ z_0>0\}.\end{aligned}$$

Therefore,

$$u(x,y,z,t)=\frac{1}{4\pi c^2 t}\left[\iint_{S_1}\psi(x_0,y_0,z_0)\,dS_0+\iint_{S_3}\psi(x_0,y_0,z_0)\,dS_0\right],\quad z>0,$$

where

$$\begin{aligned}S_1&=\{(x_0,y_0,z_0)\mid(x_0-x)^2+(y_0-y)^2+(z_0-z)^2=c^2t^2,\ z_0>0\}\\S_3&=\{(x_0,y_0,z_0)\mid(x_0-x)^2+(y_0-y)^2+(z_0+z)^2=c^2t^2,\ z_0>0\}.\end{aligned}$$

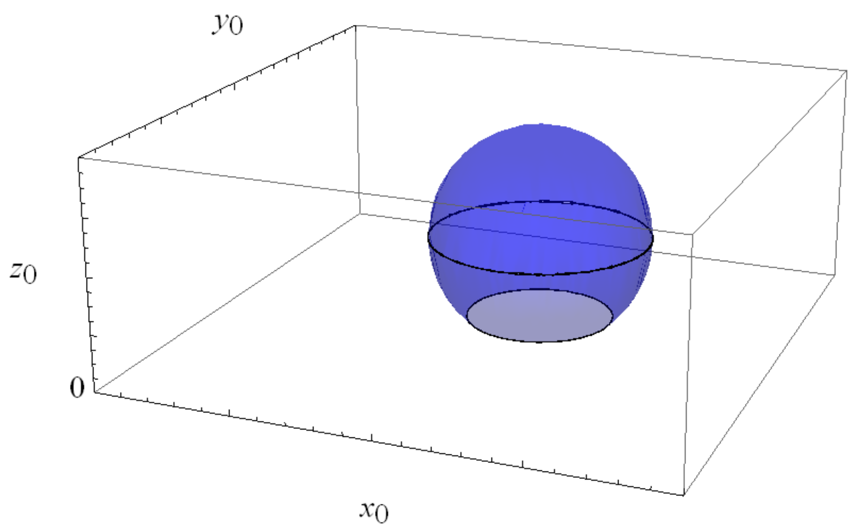

Figure 1: This figure illustrates S_1; it is the part of the sphere centered at (x,y,z) with radius ct that lies above $z_0=0$.

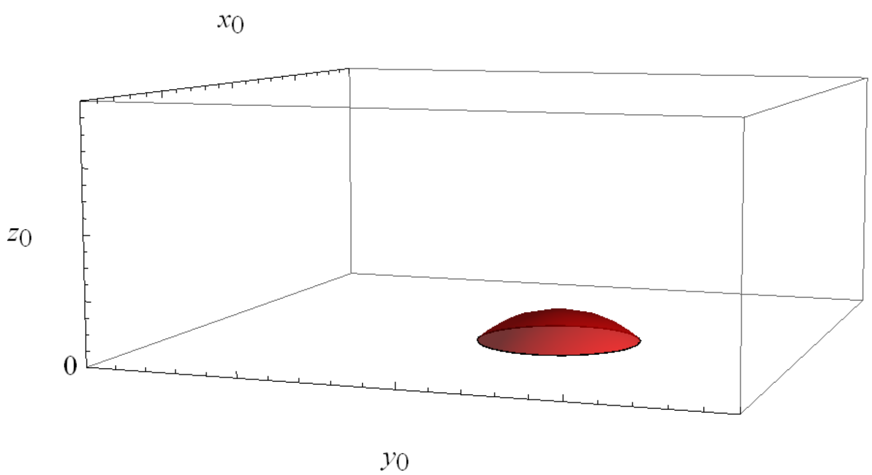

Figure 2: This figure illustrates S_3; it is the part of S_1 that has been chopped off and flipped over the $z_0 = 0$ plane. It can also be thought of as the part of the sphere centered at $(x, y, -z)$ with radius ct that lies above $z_0 = 0$.