Exercise 15

Obtain the general solution formula (19) in two dimensions from the special case (18).

Solution

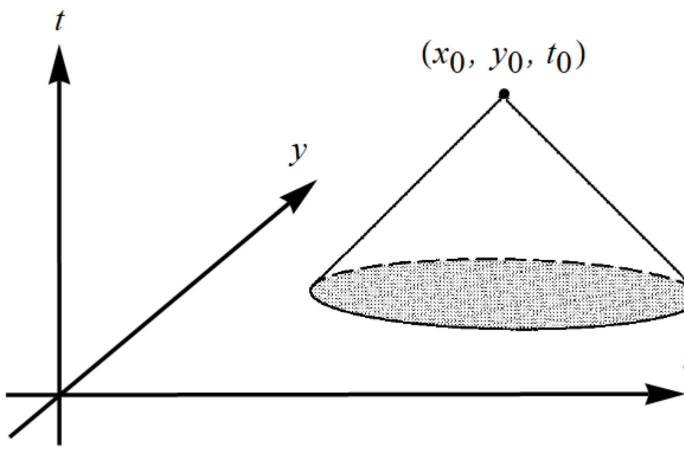
The special case in equation (18) is

$$u(0,0,t) = \frac{1}{2\pi c} \iint_{x^2+y^2 \le c^2t^2} \frac{\psi(x,y)}{(c^2t^2 - x^2 - y^2)^{1/2}}\, dx\, dy. \tag{18}$$

At a particular time $t = t_0$ these double integrals are over a disk in the xy-plane centered at $(0,0)$ with radius ct_0.

$$u(0,0,t_0) = \frac{1}{2\pi c} \iint_{x^2+y^2 \le c^2t_0^2} \frac{\psi(x,y)}{(c^2t_0^2 - x^2 - y^2)^{1/2}}\, dx\, dy$$

This is the solution of the wave equation at the origin of the xy-plane. Now we aim to find the solution at a particular point in space-time (x_0, y_0, t_0).



The wave equation is invariant to translations in space, so if $u(x,y,t)$ is a solution to the wave equation

$$\begin{aligned} &u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y < \infty,\ t > 0 \\ &u(x,y,0) = 0 \\ &u_t(x,y,0) = \psi(x,y), \end{aligned}$$

then $u(x + x_0, y + y_0, t)$ is also a solution to the wave equation, albeit with different initial conditions.

$$\begin{aligned} &u_{tt} = c^2\nabla^2 u, \quad -\infty < x, y < \infty,\ t > 0 \\ &u(x,y,0) = 0 \\ &u_t(x + x_0, y + y_0, 0) = \psi(x + x_0, y + y_0) \end{aligned}$$

Since the solution to the wave equation is unique, $u(x_0, y_0, t_0)$ has the same form as $u(0,0,t_0)$.

$$u(x_0,y_0,t_0) = \frac{1}{2\pi c} \iint_{x^2+y^2 \le c^2t_0^2} \frac{\psi(x + x_0, y + y_0)}{(c^2t_0^2 - x^2 - y^2)^{1/2}}\, dx\, dy$$

Let $k = x + x_0$ and $l = y + y_0$. Then $x = k - x_0$ and $y = l - y_0$.

$$u(x_0, y_0, t_0) = \frac{1}{2\pi c} \iint\limits_{\substack{(k-x_0)^2+(l-y_0)^2 \\ \leq c^2 t_0^2}} \frac{\psi(k, l)}{\left[c^2 t_0^2 - (k - x_0)^2 - (l - y_0)^2\right]^{1/2}} \, dk \, dl$$

k and l are dummy integration variables, so they can be replaced with x and y. Therefore,

$$u(x_0, y_0, t_0) = \frac{1}{2\pi c} \iint\limits_{\substack{(x-x_0)^2+(y-y_0)^2 \\ \leq c^2 t_0^2}} \frac{\psi(x, y)}{\left[c^2 t_0^2 - (x - x_0)^2 - (y - y_0)^2\right]^{1/2}} \, dx \, dy. \tag{19}$$

Another Way to Solve for the General Solution

The solution to the three-dimensional wave equation subject to two initial conditions,

$$\begin{aligned} &U_{tt} = c^2 \nabla^2 U, \quad -\infty < x, y, z < \infty, \ t > 0 \\ &U(x, y, z, 0) = a(x, y, z) \\ &U_t(x, y, z, 0) = b(x, y, z), \end{aligned}$$

is

$$U(x, y, z, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} a(x_0, y_0, z_0) \, dS_0\right] + \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2=c^2t^2}} b(x_0, y_0, z_0) \, dS_0.$$

Now we want to solve the corresponding problem in two dimensions,

$$\begin{aligned} &u_{tt} = c^2 \nabla^2 u, \quad -\infty < x, y < \infty, \ t > 0 \\ &u(x, y, 0) = \alpha(x, y) \\ &u_t(x, y, 0) = \beta(x, y), \end{aligned}$$

by using the method of descent. If we set $z = 0$ (or any other constant for that matter) in the formula for U, then it automatically satisfies the two-dimensional wave equation

$$\begin{aligned} \frac{\partial^2}{\partial t^2} U(x, y, 0, t) &= c^2 \left[\frac{\partial^2}{\partial x^2} U(x, y, 0, t) + \frac{\partial^2}{\partial y^2} U(x, y, 0, t) + \underbrace{\frac{\partial^2}{\partial z^2} U(x, y, 0, t)}_{=0}\right] \\ U_{tt}(x, y, 0, t) &= c^2 [U_{xx}(x, y, 0, t) + U_{yy}(x, y, 0, t)] \end{aligned}$$

and the initial conditions.

$$\begin{aligned} U(x, y, 0, 0) &= a(x, y, 0) \\ U_t(x, y, 0, 0) &= b(x, y, 0) \end{aligned}$$

Because the solution to the two-dimensional problem is unique, $U(x, y, 0, t)$ and $u(x, y, t)$ must be one and the same function. It follows that $a(x, y, 0) = \alpha(x, y)$ and $b(x, y, 0) = \beta(x, y)$. $U(x, y, 0, t)$ will now be evaluated.

$$U(x, y, 0, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +z_0^2=c^2t^2}} a(x_0, y_0, z_0) \, dS_0\right] + \frac{1}{4\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +z_0^2=c^2t^2}} b(x_0, y_0, z_0) \, dS_0$$

$(x_0 - x)^2 + (y_0 - y)^2 + z_0^2 = c^2t^2$ is the equation of a sphere in $x_0y_0z_0$-space centered at $(x, y, 0)$ with radius ct. Use the following parameterization for it (as opposed to one in spherical coordinates).

$$\begin{aligned} x_0 &= x_0 \\ y_0 &= y_0 \\ z_0 &= \sqrt{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2} \end{aligned}$$

This is actually the parameterization for the top hemisphere. Due to symmetry, the surface integrals over the sphere are double the integrals over this hemisphere's projection on the x_0y_0-plane.

$$\begin{aligned} U(x, y, 0, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t} \times 2 \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ \leq c^2t^2}} a(x_0, y_0, 0)\sqrt{1 + \left(\frac{\partial z_0}{\partial x_0}\right)^2 + \left(\frac{\partial z_0}{\partial y_0}\right)^2}\, dA_0\right] \\ + \frac{1}{4\pi c^2 t} \times 2 \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ \leq c^2t^2}} b(x_0, y_0, 0)\sqrt{1 + \left(\frac{\partial z_0}{\partial x_0}\right)^2 + \left(\frac{\partial z_0}{\partial y_0}\right)^2}\, dA_0 \end{aligned}$$

Find the partial derivatives of z_0.

$$\begin{aligned} \frac{\partial z_0}{\partial x_0} &= \frac{1}{2}\left[c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2\right]^{-1/2}[-2(x_0 - x)] = -\frac{x_0 - x}{\sqrt{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}} \\ \frac{\partial z_0}{\partial y_0} &= \frac{1}{2}\left[c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2\right]^{-1/2}[-2(y_0 - y)] = -\frac{y_0 - y}{\sqrt{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}} \end{aligned}$$

The square root becomes

$$\begin{aligned} \sqrt{1 + \left(\frac{\partial z_0}{\partial x_0}\right)^2 + \left(\frac{\partial z_0}{\partial y_0}\right)^2} &= \sqrt{1 + \frac{(x_0 - x)^2}{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2} + \frac{(y_0 - y)^2}{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}} \\ &= \sqrt{\frac{c^2t^2}{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}} \\ &= \frac{ct}{\sqrt{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}}. \end{aligned}$$

Substitute this result and replace dA_0 with $dx_0\, dy_0$ in the formula for $U(x, y, 0, t)$.

$$\begin{aligned} U(x, y, 0, t) = \frac{\partial}{\partial t}\left[\frac{1}{2\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ \leq c^2t^2}} a(x_0, y_0, 0)\frac{ct}{\sqrt{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}}\, dx_0\, dy_0\right] \\ + \frac{1}{2\pi c^2 t} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ \leq c^2t^2}} b(x_0, y_0, 0)\frac{ct}{\sqrt{c^2t^2 - (x_0 - x)^2 - (y_0 - y)^2}}\, dx_0\, dy_0 \end{aligned}$$

Bring ct in front of the integrals.

$$U(x,y,0,t) = \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\ \leq c^2t^2}} \frac{a(x_0,y_0,0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dx_0\,dy_0\right] + \frac{1}{2\pi c}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\ \leq c^2t^2}} \frac{b(x_0,y_0,0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dx_0\,dy_0$$

Therefore, replacing $U(x,y,0,t)$ with $u(x,y,t)$ and $a(x_0,y_0,0)$ with $\alpha(x_0,y_0)$ and $b(x_0,y_0,0)$ with $\beta(x_0,y_0)$, the solution to the two-dimensional initial value problem is

$$u(x,y,t) = \frac{\partial}{\partial t}\left[\frac{1}{2\pi c}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\ \leq c^2t^2}} \frac{\alpha(x_0,y_0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dx_0\,dy_0\right] + \frac{1}{2\pi c}\iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2\\ \leq c^2t^2}} \frac{\beta(x_0,y_0)}{\sqrt{c^2t^2-(x_0-x)^2-(y_0-y)^2}}\,dx_0\,dy_0.$$

These double integrals are over a disk in the x_0y_0-plane centered at (x,y) with radius ct.