Exercise 8

Carry out the passage from (11) to (13) more explicitly using spherical coordinates.

Solution

As established in Exercise 6, the unique solution to the initial value problem,

$$\begin{gathered} u_{tt} - c^2 \Delta u = f(\mathbf{x}, t) \\ u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = 0, \end{gathered}$$

is equation (11).

$$\begin{aligned} u(\mathbf{x}, t) &= \int_0^t \mathscr{S}(t-s) f(\mathbf{x}, s)\, ds \qquad (11) \\ &= \int_0^t \left[\frac{1}{4\pi c^2 (t-s)} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2 = c^2(t-s)^2}} (\cdot)\, dS_0 \right] f(\mathbf{x}, s)\, ds \\ &= \int_0^t \left[\frac{1}{4\pi c^2 (t-s)} \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2 = c^2(t-s)^2}} f(\mathbf{x}_0, s)\, dS_0 \right] ds \\ &= \frac{1}{4\pi c^2} \int_0^t \iint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2 = c^2(t-s)^2}} \frac{f(x_0, y_0, z_0, s)}{t-s}\, dS_0\, ds \end{aligned}$$

Write the surface integral over the sphere centered at (x, y, z) with radius $c(t-s)$ explicitly by using spherical coordinates (r_0, θ_0, ϕ_0), where ϕ_0 represents the angle from the polar axis.

$$\begin{aligned} x_0 - x &= c(t-s) \sin\theta_0 \cos\phi_0 \\ y_0 - y &= c(t-s) \sin\theta_0 \sin\phi_0 \\ z_0 - z &= c(t-s) \cos\theta_0 \end{aligned}$$

The solution becomes

$$\begin{aligned} u(x, y, z, t) &= \frac{1}{4\pi c^2} \int_0^t \int_0^\pi \int_0^{2\pi} \frac{f(c(t-s), \theta_0, \phi_0, s)}{t-s} [c^2 (t-s)^2 \sin\phi_0\, d\theta_0\, d\phi_0]\, ds \\ &= \frac{1}{4\pi c^2} \int_0^t \int_0^\pi \int_0^{2\pi} f(c(t-s), \theta_0, \phi_0, s) [c^2 (t-s) \sin\phi_0\, d\theta_0\, d\phi_0]\, ds. \end{aligned}$$

Make the following substitution.

$$\begin{aligned} r_0 &= c(t-s) \quad \rightarrow \quad s = t - \frac{r_0}{c} \\ dr_0 &= -c\, ds \end{aligned}$$

As a result,

$$\begin{aligned}
u(x,y,z,t) &= \frac{1}{4\pi c^2}\int_{ct}^{0}\int_0^{\pi}\int_0^{2\pi} f\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)(r_0\sin\phi_0\,d\theta_0\,d\phi_0)(-dr_0) \\
&= \frac{1}{4\pi c^2}\int_0^{ct}\int_0^{\pi}\int_0^{2\pi} f\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) r_0\sin\phi_0\,d\theta_0\,d\phi_0\,dr_0 \\
&= \frac{1}{4\pi c^2}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct} f\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right) r_0\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0 \\
&= \frac{1}{4\pi c^2}\int_0^{\pi}\int_0^{2\pi}\int_0^{ct} \frac{f\left(r_0,\theta_0,\phi_0,t-\frac{r_0}{c}\right)}{r_0} r_0^2\sin\phi_0\,dr_0\,d\theta_0\,d\phi_0.
\end{aligned}$$

Note that

$$(x_0-x)^2+(y_0-y)^2+(z_0-z)^2 = r_0^2 \quad \to \quad r_0 = \sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}.$$

Changing back to Cartesian coordinates, we have

$$u(x,y,z,t) = \frac{1}{4\pi c^2} \iiint\limits_{\substack{(x_0-x)^2+(y_0-y)^2 \\ +(z_0-z)^2\le c^2t^2}} \frac{f\left(x_0,y_0,z_0,t-\frac{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}{c}\right)}{\sqrt{(x_0-x)^2+(y_0-y)^2+(z_0-z)^2}}\,dV_0,$$

which is equation (13).

$$u(\mathbf{x},t) = \frac{1}{4\pi c^2}\iiint\limits_{\{|\boldsymbol{\xi}-\mathbf{x}|\le ct\}} \frac{f(\boldsymbol{\xi},t-|\boldsymbol{\xi}-\mathbf{x}|/c)}{|\boldsymbol{\xi}-\mathbf{x}|}\,d\boldsymbol{\xi}. \tag{13}$$