Exercise 3

Find the solution of the diffusion equation in the half-space $\{(x, y, z, t) : z > 0\}$ with the Neumann condition $\partial u/\partial z = 0$ on $z = 0$. (*Hint:* Use the method of reflection.)

Solution

Here we have to solve

$$\begin{gathered} u_t = \kappa \nabla^2 u, \quad -\infty < x, y < \infty,\ 0 < z < \infty,\ t > 0 \\ u(x, y, z, 0) = \phi(x, y, z) \\ u_z(x, y, 0, t) = 0. \end{gathered}$$

Since z is over a semi-infinite interval, we can use the method of reflection to find the solution. Consider the same problem over an infinite interval, using the even extension of ϕ in order to satisfy the Neumann boundary condition at $z = 0$:

$$\begin{gathered} w_t = \kappa \nabla^2 w, \quad -\infty < x, y, z < \infty,\ t > 0 \\ w(x, y, z, 0) = \phi_{\text{even}}(x, y, z), \end{gathered}$$

where

$$\phi_{\text{even}}(x, y, z) = \begin{cases} \phi(x, y, z) & \text{if } z > 0 \\ \phi(x, y, -z) & \text{if } z < 0 \end{cases}.$$

The solution for w is given by

$$w(x, y, z, t) = \frac{1}{(4\pi\kappa t)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-k)^2 + (y-l)^2 + (z-m)^2}{4\kappa t}\right] \phi_{\text{even}}(k, l, m)\, dk\, dl\, dm.$$

The solution for u is then just the restriction of w to $z > 0$.

$$u(x, y, z, t) = \frac{1}{(4\pi\kappa t)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \exp\left[-\frac{(x-k)^2 + (y-l)^2 + (z-m)^2}{4\kappa t}\right] \phi_{\text{even}}(k, l, m)\, dk\, dl\, dm, \quad z > 0$$

Our task now is to write this formula in terms of the given function $\phi(x, y, z)$. Split up the integral in dm into two parts, one over the negative values of m and one over the positive values of m.

$$\begin{aligned} u(x, y, z, t) = \frac{1}{(4\pi\kappa t)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bigg\{ \int_{-\infty}^{0} \exp\left[-\frac{(x-k)^2 + (y-l)^2 + (z-m)^2}{4\kappa t}\right] \phi(k, l, -m)\, dm \\ + \int_{0}^{\infty} \exp\left[-\frac{(x-k)^2 + (y-l)^2 + (z-m)^2}{4\kappa t}\right] \phi(k, l, m)\, dm \bigg\}\, dk\, dl, \quad z > 0 \end{aligned}$$

Substitute $n = -m$ in the first integral and $n = m$ in the second integral.

$$\begin{aligned} u(x, y, z, t) = \frac{1}{(4\pi\kappa t)^{3/2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \bigg\{ \int_{\infty}^{0} \exp\left[-\frac{(x-k)^2 + (y-l)^2 + (z+n)^2}{4\kappa t}\right] \phi(k, l, n)(-dn) \\ + \int_{0}^{\infty} \exp\left[-\frac{(x-k)^2 + (y-l)^2 + (z-n)^2}{4\kappa t}\right] \phi(k, l, n)\, dn \bigg\}\, dk\, dl, \quad z > 0 \end{aligned}$$

Use the minus sign in front of dn to switch the limits of integration.

$$u(x,y,z,t) = \frac{1}{(4\pi\kappa t)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \int_0^{\infty} \exp\left[-\frac{(x-k)^2+(y-l)^2+(z+n)^2}{4\kappa t}\right] \phi(k,l,n)\,dn + \int_0^{\infty} \exp\left[-\frac{(x-k)^2+(y-l)^2+(z-n)^2}{4\kappa t}\right] \phi(k,l,n)\,dn \right\} dk\,dl, \quad z>0$$

Combine the integrals.

$$u(x,y,z,t) = \frac{1}{(4\pi\kappa t)^{3/2}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_0^{\infty} \left\{ \exp\left[-\frac{(x-k)^2+(y-l)^2+(z+n)^2}{4\kappa t}\right] + \exp\left[-\frac{(x-k)^2+(y-l)^2+(z-n)^2}{4\kappa t}\right] \right\} \phi(k,l,n)\,dn\,dk\,dl, \quad z>0$$

Therefore,

$$u(x,y,z,t) = \frac{1}{(4\pi\kappa t)^{3/2}} \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{ \exp\left[-\frac{(x-k)^2+(y-l)^2+(z+n)^2}{4\kappa t}\right] + \exp\left[-\frac{(x-k)^2+(y-l)^2+(z-n)^2}{4\kappa t}\right] \right\} \phi(k,l,n)\,dk\,dl\,dn, \quad z>0.$$

This solution can be written compactly as

$$u(x,y,z,t) = \int_0^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [G_3(x-k,y-l,z+n,t) + G_3(x-k,y-l,z-n,t)]\phi(k,l,n)\,dk\,dl\,dn,$$

where

$$G_3(x,y,z,t) = \frac{1}{(4\pi\kappa t)^{3/2}} \exp\left(-\frac{x^2+y^2+z^2}{4\kappa t}\right)$$

is the Green's function for the three-dimensional diffusion equation.