Problem 1.10

A particle moves in a circle (center O and radius R) with constant angular velocity ω counterclockwise. The circle lies in the xy plane and the particle is on the x axis at time $t = 0$. Show that the particle's position is given by

$$\mathbf{r}(t) = \hat{\mathbf{x}} R\cos(\omega t) + \hat{\mathbf{y}} R\sin(\omega t).$$

Find the particle's velocity and acceleration. What are the magnitude and direction of the acceleration? Relate your results to well-known properties of uniform circular motion.

Solution

The equation for a circle in the xy-plane centered at the origin with radius R is

$$x^2 + y^2 = R^2.$$

The components of the particle's motion, $x = x(t)$ and $y = y(t)$, have to satisfy this equation. They are expected to be in terms of sine and cosine. To know specifically, use information about the particle's position.

$$\begin{aligned} \omega t = 0: &\quad x = R \quad \text{and} \quad y = 0 \\ \omega t = \frac{\pi}{2}: &\quad x = 0 \quad \text{and} \quad y = R \end{aligned}$$

Because x starts at a maximum and then decreases, use $x(t) = R\cos\omega t$, and because y starts at zero and increases, use $y(t) = R\sin\omega t$. Therefore, the position vector is

$$\begin{aligned} \mathbf{r}(t) &= \langle x(t), y(t)\rangle \\ &= \langle R\cos\omega t, R\sin\omega t\rangle \\ &= \hat{\mathbf{x}} R\cos(\omega t) + \hat{\mathbf{y}} R\sin(\omega t). \end{aligned}$$

The velocity vector is obtained by differentiating the position vector with respect to time.

$$\begin{aligned} \mathbf{v}(t) &= \mathbf{r}'(t) \\ &= \langle x'(t), y'(t)\rangle \\ &= \langle -R\omega\sin\omega t, R\omega\cos\omega t\rangle \end{aligned}$$

Note that the magnitude of $\mathbf{v}$, or the speed that the particle moves around the circle, is

$$|\mathbf{v}| = \sqrt{(-R\omega\sin\omega t)^2 + (R\omega\cos\omega t)^2} = R\omega.$$

The acceleration vector is obtained by differentiating the velocity vector with respect to time.

$$\begin{aligned} \mathbf{a}(t) &= \mathbf{v}'(t) = \mathbf{r}''(t) \\ &= \langle x''(t), y''(t)\rangle \\ &= \langle -R\omega^2\cos\omega t, -R\omega^2\sin\omega t\rangle \end{aligned}$$

Note that the magnitude of $\mathbf{a}$, or the centripetal acceleration, is

$$|\mathbf{a}| = \sqrt{(-R\omega^2\cos\omega t)^2 + (-R\omega^2\sin\omega t)^2} = R\omega^2.$$