Problem 1.12

The position of a moving particle is given as a function of time t to be

$$\mathbf{r}(t) = \hat{\mathbf{x}} b \cos(\omega t) + \hat{\mathbf{y}} c \sin(\omega t) + \hat{\mathbf{z}} v_0 t,$$

where b, c, v_0 and ω are constants. Describe the particle's orbit.

Solution

Write the given position vector as

$$\begin{aligned} \mathbf{r}(t) &= \langle b \cos \omega t, c \sin \omega t, v_0 t \rangle \\ &= \langle x(t), y(t), z(t) \rangle. \end{aligned}$$

To determine the particle's orbit, it's necessary to find the equation satisfied by x and y. Start with a known trigonometric identity involving $\cos \omega t$ and $\sin \omega t$.

$$\cos^2 \omega t + \sin^2 \omega t = 1$$

$$\frac{b^2 \cos^2 \omega t}{b^2} + \frac{c^2 \sin^2 \omega t}{c^2} = 1$$

$$\frac{x^2}{b^2} + \frac{y^2}{c^2} = 1$$

This is the equation for an ellipse, so the particle's orbit perpendicular to the z-axis is elliptical. The component of motion parallel to the z-axis is $v_0 t$, which means the height increases v_0 units every second. The particle's orbit is therefore an elliptical helix that starts at

$$\mathbf{r}(0) = \langle b, 0, 0 \rangle.$$