Problem 1.16

(a) Defining the scalar product $\mathbf{r} \cdot \mathbf{s}$ by Equation (1.7), $\mathbf{r} \cdot \mathbf{s} = \sum r_i s_i$, show that Pythagoras's theorem implies that the magnitude of any vector $\mathbf{r}$ is $r = \sqrt{\mathbf{r} \cdot \mathbf{r}}$. **(b)** It is clear that the length of a vector does not depend on our choice of coordinate axes. Thus the result of part (a) guarantees that the scalar product $\mathbf{r} \cdot \mathbf{r}$, as defined by (1.7), is the same for any choice of orthogonal axes. Use this to prove that $\mathbf{r} \cdot \mathbf{s}$, as defined by (1.7), is the same for any choice of orthogonal axes. [*Hint:* Consider the length of the vector $\mathbf{r} + \mathbf{s}$.]

Solution

Part (a)

Substitute $\mathbf{r}$ for $\mathbf{s}$ in the definition of the scalar product.

$$\begin{aligned}\mathbf{r} \cdot \mathbf{r} &= \sum_{i=1}^{3} r_i r_i \\ &= \sum_{i=1}^{3} r_i^2 \\ &= r_x^2 + r_y^2 + r_z^2\end{aligned}$$

Use the Pythagorean theorem on the right side.

$$\mathbf{r} \cdot \mathbf{r} = r^2$$

Therefore, taking the square root of both sides,

$$r = \sqrt{\mathbf{r} \cdot \mathbf{r}}.$$

Part (b)

Consider the length of the vector $\mathbf{r} + \mathbf{s}$.

$$\begin{aligned}|\mathbf{r} + \mathbf{s}| &= \sqrt{(r_x + s_x)^2 + (r_y + s_y)^2 + (r_z + s_z)^2} \\ &= \sqrt{(r_x^2 + 2r_x s_x + s_x^2) + (r_y^2 + 2r_y s_y + s_y^2) + (r_z^2 + 2r_z s_z + s_z^2)} \\ &= \sqrt{(r_x^2 + r_y^2 + r_z^2) + (s_x^2 + s_y^2 + s_z^2) + 2(r_x s_x + r_y s_y + r_z s_z)} \\ &= \sqrt{r^2 + s^2 + 2(\mathbf{r} \cdot \mathbf{s})}\end{aligned}$$

Square both sides.

$$|\mathbf{r} + \mathbf{s}|^2 = r^2 + s^2 + 2(\mathbf{r} \cdot \mathbf{s})$$

Solve for $\mathbf{r} \cdot \mathbf{s}$.

$$\mathbf{r} \cdot \mathbf{s} = \frac{1}{2}\left(|\mathbf{r} + \mathbf{s}|^2 - r^2 - s^2\right)$$

Since $\mathbf{r} \cdot \mathbf{s}$ is written in terms of lengths of vectors, which are known to be independent of the choice of axes, the dot product is independent of the choice of axes.