Problem 1.17

(a) Prove that the vector product $\mathbf{r} \times \mathbf{s}$ as defined by (1.9) is distributive; that is, that $\mathbf{r} \times (\mathbf{u} + \mathbf{v}) = (\mathbf{r} \times \mathbf{u}) + (\mathbf{r} \times \mathbf{v})$. **(b)** Prove the product rule

$$\frac{d}{dt}(\mathbf{r} \times \mathbf{s}) = \mathbf{r} \times \frac{d\mathbf{s}}{dt} + \frac{d\mathbf{r}}{dt} \times \mathbf{s}.$$

Be careful with the order of the factors.

Solution

According to equation (1.9), the vector product $\mathbf{r} \times \mathbf{s}$ is defined as

$$\mathbf{r} \times \mathbf{s} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ s_x & s_y & s_z \end{vmatrix} = (r_y s_z - r_z s_y)\hat{\mathbf{x}} - (r_x s_z - r_z s_x)\hat{\mathbf{y}} + (r_x s_y - r_y s_x)\hat{\mathbf{z}}. \tag{1.9}$$

Part (a)

Replace $\mathbf{s}$ with $\mathbf{u} + \mathbf{v}$ in the definition and simplify.

$$\begin{aligned}
\mathbf{r} \times (\mathbf{u} + \mathbf{v}) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ r_x & r_y & r_z \\ u_x + v_x & u_y + v_y & u_z + v_z \end{vmatrix} \\
&= [r_y(u_z + v_z) - r_z(u_y + v_y)]\hat{\mathbf{x}} - [r_x(u_z + v_z) - r_z(u_x + v_x)]\hat{\mathbf{y}} + [r_x(u_y + v_y) - r_y(u_x + v_x)]\hat{\mathbf{z}} \\
&= [(r_y u_z - r_z u_y) + (r_y v_z - r_z v_y)]\hat{\mathbf{x}} - [(r_x u_z - r_z u_x) + (r_x v_z - r_z v_x)]\hat{\mathbf{y}} \\
&\qquad + [(r_x u_y - r_y u_x) + (r_x v_y - r_y v_x)]\hat{\mathbf{z}} \\
&= [(r_y u_z - r_z u_y)\hat{\mathbf{x}} - (r_x u_z - r_z u_x)\hat{\mathbf{y}} + (r_x u_y - r_y u_x)\hat{\mathbf{z}}] \\
&\qquad + [(r_y v_z - r_z v_y)\hat{\mathbf{x}} - (r_x v_z - r_z v_x)\hat{\mathbf{y}} + (r_x v_y - r_y v_x)\hat{\mathbf{z}}] \\
&= \mathbf{r} \times \mathbf{u} + \mathbf{r} \times \mathbf{v}
\end{aligned}$$

Therefore, the cross product is distributive.

Part (b)

Differentiate both sides of equation (1.9) with respect to t.

$$\begin{aligned}
\frac{d}{dt}(\mathbf{r}\times\mathbf{s}) &= \frac{d}{dt}\left[(r_y s_z - r_z s_y)\hat{\mathbf{x}} - (r_x s_z - r_z s_x)\hat{\mathbf{y}} + (r_x s_y - r_y s_x)\hat{\mathbf{z}}\right] \\
&= \frac{d}{dt}(r_y s_z - r_z s_y)\hat{\mathbf{x}} - \frac{d}{dt}(r_x s_z - r_z s_x)\hat{\mathbf{y}} + \frac{d}{dt}(r_x s_y - r_y s_x)\hat{\mathbf{z}} \\
&= \left[\frac{d}{dt}(r_y s_z) - \frac{d}{dt}(r_z s_y)\right]\hat{\mathbf{x}} - \left[\frac{d}{dt}(r_x s_z) - \frac{d}{dt}(r_z s_x)\right]\hat{\mathbf{y}} + \left[\frac{d}{dt}(r_x s_y) - \frac{d}{dt}(r_y s_x)\right]\hat{\mathbf{z}} \\
&= \left(\frac{dr_y}{dt}s_z + r_y\frac{ds_z}{dt} - \frac{dr_z}{dt}s_y - r_z\frac{ds_y}{dt}\right)\hat{\mathbf{x}} - \left(\frac{dr_x}{dt}s_z + r_x\frac{ds_z}{dt} - \frac{dr_z}{dt}s_x - r_z\frac{ds_x}{dt}\right)\hat{\mathbf{y}} \\
&\qquad + \left(\frac{dr_x}{dt}s_y + r_x\frac{ds_y}{dt} - \frac{dr_y}{dt}s_x - r_y\frac{ds_x}{dt}\right)\hat{\mathbf{z}} \\
&= \left[\left(\frac{dr_y}{dt}s_z - \frac{dr_z}{dt}s_y\right)\hat{\mathbf{x}} - \left(\frac{dr_x}{dt}s_z - \frac{dr_z}{dt}s_x\right)\hat{\mathbf{y}} + \left(\frac{dr_x}{dt}s_y - \frac{dr_y}{dt}s_x\right)\hat{\mathbf{z}}\right] \\
&\qquad + \left[\left(r_y\frac{ds_z}{dt} - r_z\frac{ds_y}{dt}\right)\hat{\mathbf{x}} - \left(r_x\frac{ds_z}{dt} - r_z\frac{ds_x}{dt}\right)\hat{\mathbf{y}} + \left(r_x\frac{ds_y}{dt} - r_y\frac{ds_x}{dt}\right)\hat{\mathbf{z}}\right] \\
&= \frac{d\mathbf{r}}{dt}\times\mathbf{s} + \mathbf{r}\times\frac{d\mathbf{s}}{dt}
\end{aligned}$$

Therefore, the product rule can be applied when differentiating a cross product with respect to time.