Problem 1.19

If $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$ denote the position, velocity, and acceleration of a particle, prove that

$$\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] = \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}).$$

Solution

Using the results from Problem 1.8 and Problem 1.17,

$$\begin{aligned}\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) &= \frac{d\mathbf{r}}{dt}\cdot\mathbf{s} + \mathbf{r}\cdot\frac{d\mathbf{s}}{dt} \\ \frac{d}{dt}(\mathbf{r}\times\mathbf{s}) &= \frac{d\mathbf{r}}{dt}\times\mathbf{s} + \mathbf{r}\times\frac{d\mathbf{s}}{dt},\end{aligned}$$

start with the left side and simplify the result.

$$\begin{aligned}\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] &= \frac{d\mathbf{a}}{dt}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\frac{d}{dt}(\mathbf{v}\times\mathbf{r}) \\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot\left(\frac{d\mathbf{v}}{dt}\times\mathbf{r} + \mathbf{v}\times\frac{d\mathbf{r}}{dt}\right) \\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot(\mathbf{a}\times\mathbf{r} + \underbrace{\mathbf{v}\times\mathbf{v}}_{=\mathbf{0}}) \\ &= \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}) + \mathbf{a}\cdot(\mathbf{a}\times\mathbf{r})\end{aligned}$$

The vector $\mathbf{a}\times\mathbf{r}$ is perpendicular to $\mathbf{a}$, which means the dot product of $\mathbf{a}$ and $\mathbf{a}\times\mathbf{r}$ is zero: $\mathbf{a}\cdot(\mathbf{a}\times\mathbf{r}) = 0$. Therefore,

$$\frac{d}{dt}[\mathbf{a}\cdot(\mathbf{v}\times\mathbf{r})] = \dot{\mathbf{a}}\cdot(\mathbf{v}\times\mathbf{r}).$$