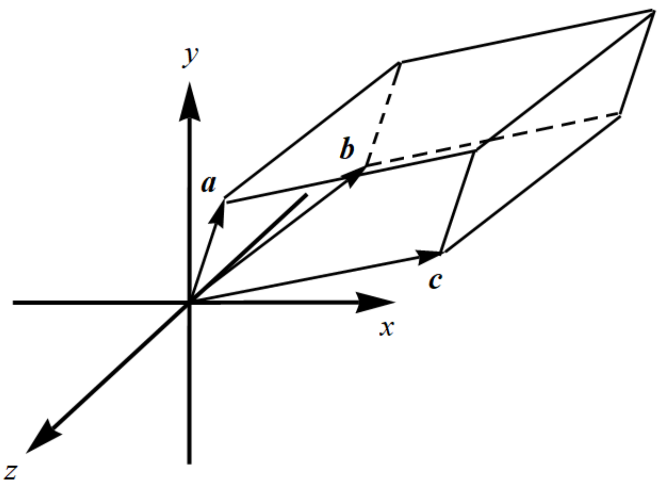
Problem 1.21

A parallelepiped (a six-faced solid with opposite faces parallel) has one corner at the origin O and the three edges that emanate from O defined by vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$. Show that the volume of the parallelepiped is $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

Solution

Draw the vectors, $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, and the parallelepiped they form.

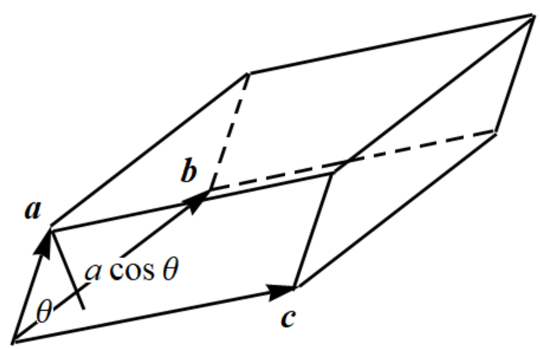


The volume of this parallelepiped is the area of one of its faces times the height in the perpendicular direction.

$$V = Ah$$

Take the area of the bottom face, a parallelogram; it's area is the magnitude of the cross product of $\mathbf{b}$ and $\mathbf{c}$.

$$A = |\mathbf{b}\times\mathbf{c}|$$



If we let θ be the angle between $\mathbf{a}$ and the plane containing $\mathbf{b}$ and $\mathbf{c}$, then the perpendicular height is $h = |\mathbf{a}|\cos\theta$. The volume of the parallelepiped, which must be positive, is then

$$\begin{aligned} V &= |\mathbf{b}\times\mathbf{c}|(|\mathbf{a}|\cos\theta) \\ &= |\mathbf{a}||\mathbf{b}\times\mathbf{c}|\cos\theta \\ &= |\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|. \end{aligned}$$