Problem 1.23

The unknown vector $\mathbf{v}$ satisfies $\mathbf{b}\cdot\mathbf{v}=\lambda$ and $\mathbf{b}\times\mathbf{v}=\mathbf{c}$, where λ, $\mathbf{b}$, and $\mathbf{c}$ are fixed and known. Find $\mathbf{v}$ in terms of λ, $\mathbf{b}$, and $\mathbf{c}$.

Solution

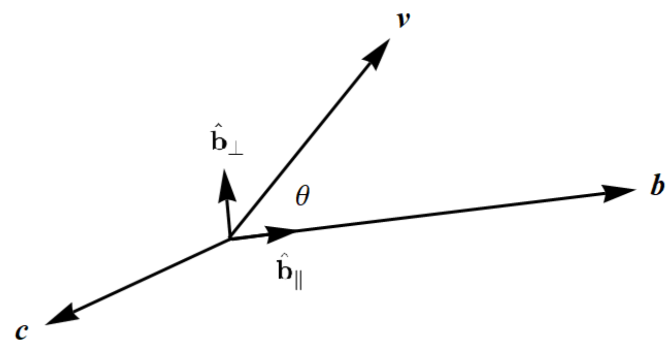
Take the magnitude of both sides of the equation involving the cross product.

$$\mathbf{b}\cdot\mathbf{v}=\lambda$$
$$|\mathbf{b}\times\mathbf{v}|=|\mathbf{c}|$$

Suppose that θ is the angle between $\mathbf{b}$ and $\mathbf{v}$.

$$|\mathbf{b}||\mathbf{v}|\cos\theta=\lambda$$
$$|\mathbf{b}||\mathbf{v}|\sin\theta=|\mathbf{c}|$$

From the equation $\mathbf{b}\times\mathbf{v}=\mathbf{c}$, the vector $\mathbf{c}$ is perpendicular to the plane containing $\mathbf{b}$ and $\mathbf{v}$.



A basis for this plane can be written in terms of $\hat{\mathbf{b}}_{\|}$, a unit vector parallel to $\mathbf{b}$, and $\hat{\mathbf{b}}_{\perp}$, a unit vector perpendicular to $\mathbf{b}$ that also lies in the plane.

$$\hat{\mathbf{b}}_{\|}=\frac{\mathbf{b}}{|\mathbf{b}|}\qquad\qquad \hat{\mathbf{b}}_{\perp}=\frac{\mathbf{c}\times\mathbf{b}}{|\mathbf{c}\times\mathbf{b}|}$$

Write $\mathbf{v}$ in terms of these basis vectors.

$$\mathbf{v}=v_1\hat{\mathbf{b}}_{\|}+v_2\hat{\mathbf{b}}_{\perp}$$

To determine v_1, take the dot product of both sides with $\hat{\mathbf{b}}_{\|}$.

$$\mathbf{v}\cdot\hat{\mathbf{b}}_{\|}=v_1(\hat{\mathbf{b}}_{\|}\cdot\hat{\mathbf{b}}_{\|})+v_2(\hat{\mathbf{b}}_{\perp}\cdot\hat{\mathbf{b}}_{\|})$$
$$|\mathbf{v}|\cos\theta=v_1(1)+v_2(0)$$
$$\frac{\lambda}{|\mathbf{b}|}=v_1$$

To determine v_2, take the dot product of both sides with $\hat{\mathbf{b}}_{\perp}$.

$$\mathbf{v}\cdot\hat{\mathbf{b}}_{\perp}=v_1(\hat{\mathbf{b}}_{\|}\cdot\hat{\mathbf{b}}_{\perp})+v_2(\hat{\mathbf{b}}_{\perp}\cdot\hat{\mathbf{b}}_{\perp})$$
$$|\mathbf{v}|\sin\theta=v_1(0)+v_2(1)$$
$$\frac{|\mathbf{c}|}{|\mathbf{b}|}=v_2$$

Therefore,

$$\begin{aligned}\mathbf{v} &= \frac{\lambda}{|\mathbf{b}|}\frac{\mathbf{b}}{|\mathbf{b}|} + \frac{|\mathbf{c}|}{|\mathbf{b}|}\frac{\mathbf{c}\times\mathbf{b}}{|\mathbf{c}\times\mathbf{b}|} \\ &= \lambda\frac{\mathbf{b}}{|\mathbf{b}|^2} + \frac{|\mathbf{c}|}{|\mathbf{b}|}\frac{\mathbf{c}\times\mathbf{b}}{|\mathbf{c}||\mathbf{b}|\sin 90^\circ} \\ &= \frac{\lambda\mathbf{b} + \mathbf{c}\times\mathbf{b}}{|\mathbf{b}|^2} \\ &= \frac{\lambda\mathbf{b} - \mathbf{b}\times\mathbf{c}}{|\mathbf{b}|^2}.\end{aligned}$$