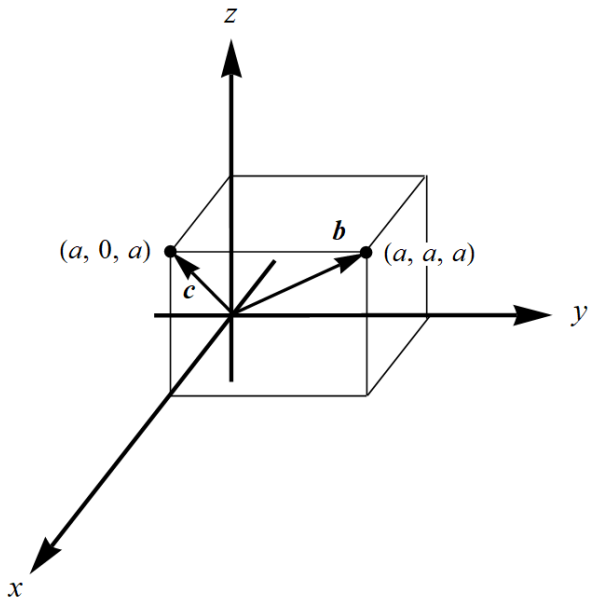
Problem 1.5

Find the angle between a body diagonal of a cube and any one of its face diagonals. [*Hint:* Choose a cube with side 1 and with one corner at O and the opposite corner at the point $(1, 1, 1)$. Write down the vector that represents a body diagonal and another that represents a face diagonal, and then find the angle between them as in Problem 1.4.]

Solution

Suppose there's a cube with sides of length a that has one of its corners at the origin. Let $\mathbf{b}$ and $\mathbf{c}$ be the body and face diagonal vectors shown below.



Use the definition of the dot product to determine the angle θ between these vectors.

$$\mathbf{b} \cdot \mathbf{c} = |\mathbf{b}||\mathbf{c}| \cos\theta$$

Solve for $\cos\theta$.

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{b} \cdot \mathbf{c}}{|\mathbf{b}||\mathbf{c}|} \\
&= \frac{\langle a, a, a\rangle \cdot \langle a, 0, a\rangle}{\sqrt{a^2 + a^2 + a^2}\sqrt{a^2 + a^2}} \\
&= \frac{(a)(a) + (a)(0) + (a)(a)}{\sqrt{3a^2}\sqrt{2a^2}} \\
&= \frac{2a^2}{\sqrt{6}a^2} \\
&= \frac{2}{\sqrt{6}}
\end{aligned}$$

Therefore, the angle between a body diagonal of a cube and any one of its face diagonals is

$$\theta = \cos^{-1}\left(\frac{2}{\sqrt{6}}\right) \approx 35.3^\circ.$$