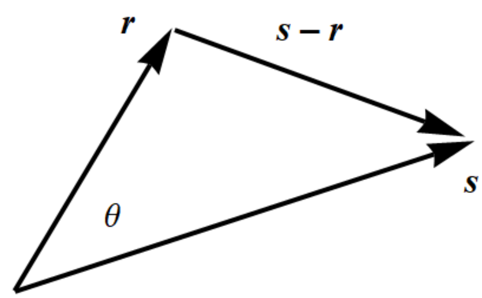
Problem 1.7

Prove that the two definitions of the scalar product $\mathbf{r} \cdot \mathbf{s}$ as $rs\cos\theta$ (1.6) and $\sum r_i s_i$ (1.7) are equal. One way to do this is to choose your x axis along the direction of $\mathbf{r}$. [Strictly speaking you should first make sure that the definition (1.7) is independent of the choice of axes. If you like to worry about such niceties, see Problem 1.16.]

Solution

Consider two arbitrary vectors, $\mathbf{r} = \langle r_x, r_y, r_z \rangle$ and $\mathbf{s} = \langle s_x, s_y, s_z \rangle$, in space that have an angle θ between them.



According to the law of cosines, the sides of this triangle are related by

$$|\mathbf{s} - \mathbf{r}|^2 = |\mathbf{r}|^2 + |\mathbf{s}|^2 - 2|\mathbf{r}||\mathbf{s}|\cos\theta.$$

Solve for $\cos\theta$.

$$\cos\theta = \frac{|\mathbf{r}|^2 + |\mathbf{s}|^2 - |\mathbf{s} - \mathbf{r}|^2}{2|\mathbf{r}||\mathbf{s}|}$$

Now consider the dot product of $\mathbf{r}$ and $\mathbf{s}$ and use this result for $\cos\theta$.

$$\begin{aligned}
\mathbf{r} \cdot \mathbf{s} &= |\mathbf{r}||\mathbf{s}|\cos\theta \\
&= |\mathbf{r}||\mathbf{s}|\left(\frac{|\mathbf{r}|^2 + |\mathbf{s}|^2 - |\mathbf{s} - \mathbf{r}|^2}{2|\mathbf{r}||\mathbf{s}|}\right) \\
&= \frac{|\mathbf{r}|^2 + |\mathbf{s}|^2 - |\mathbf{s} - \mathbf{r}|^2}{2} \\
&= \frac{\mathbf{r} \cdot \mathbf{r} + \mathbf{s} \cdot \mathbf{s} - (\mathbf{s} - \mathbf{r}) \cdot (\mathbf{s} - \mathbf{r})}{2} \\
&= \frac{\langle r_x, r_y, r_z \rangle \cdot \langle r_x, r_y, r_z \rangle + \langle s_x, s_y, s_z \rangle \cdot \langle s_x, s_y, s_z \rangle - \langle s_x - r_x, s_y - r_y, s_z - r_z \rangle \cdot \langle s_x - r_x, s_y - r_y, s_z - r_z \rangle}{2} \\
&= \frac{(r_x^2 + r_y^2 + r_z^2) + (s_x^2 + s_y^2 + s_z^2) - [(s_x - r_x)^2 + (s_y - r_y)^2 + (s_z - r_z)^2]}{2} \\
&= \frac{(r_x^2 + r_y^2 + r_z^2) + (s_x^2 + s_y^2 + s_z^2) - [(s_x^2 - 2r_x s_x + r_x^2) + (s_y^2 - 2r_y s_y + r_y^2) + (s_z^2 - 2r_z s_z + r_z^2)]}{2} \\
&= \frac{2r_x s_x + 2r_y s_y + 2r_z s_z}{2} \\
&= r_x s_x + r_y s_y + r_z s_z \\
&= \sum_{i=1}^{3} r_i s_i
\end{aligned}$$