Problem 1.8

(a) Use the definition (1.7) to prove that the scalar product is distributive, that is, $\mathbf{r}\cdot(\mathbf{u}+\mathbf{v}) = \mathbf{r}\cdot\mathbf{u}+\mathbf{r}\cdot\mathbf{v}$. **(b)** If $\mathbf{r}$ and $\mathbf{s}$ are vectors that depend on time, prove that the product rule for differentiating products applies to $\mathbf{r}\cdot\mathbf{s}$, that is, that

$$\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) = \mathbf{r}\cdot\frac{d\mathbf{s}}{dt}+\frac{d\mathbf{r}}{dt}\cdot\mathbf{s}.$$

Solution

Start with definition (1.7).

$$\mathbf{r}\cdot\mathbf{s} = \sum_{i=1}^{3} r_i s_i \tag{1.7}$$

Part (a)

Substitute $\mathbf{s} = \mathbf{u}+\mathbf{v}$ and simplify the result.

$$\begin{aligned}
\mathbf{r}\cdot(\mathbf{u}+\mathbf{v}) &= \sum_{i=1}^{3} r_i(u_i+v_i) \\
&= \sum_{i=1}^{3}(r_iu_i+r_iv_i) \\
&= \sum_{i=1}^{3} r_iu_i + \sum_{i=1}^{3} r_iv_i \\
&= \mathbf{r}\cdot\mathbf{u}+\mathbf{r}\cdot\mathbf{v}
\end{aligned}$$

Therefore, the dot product is distributive.

Part (b)

Differentiate both sides of equation (1.7) with respect to t.

$$\begin{aligned}
\frac{d}{dt}(\mathbf{r}\cdot\mathbf{s}) &= \frac{d}{dt}\sum_{i=1}^{3} r_is_i \\
&= \sum_{i=1}^{3}\frac{d}{dt}(r_is_i) \\
&= \sum_{i=1}^{3}\left(\frac{dr_i}{dt}s_i + r_i\frac{ds_i}{dt}\right) \\
&= \sum_{i=1}^{3}\frac{dr_i}{dt}s_i + \sum_{i=1}^{3} r_i\frac{ds_i}{dt} \\
&= \frac{d\mathbf{r}}{dt}\cdot\mathbf{s}+\mathbf{r}\cdot\frac{d\mathbf{s}}{dt}
\end{aligned}$$

Therefore, the product rule works for the dot product.