Problem 1.13

Determine the rms value of a wave consisting of the positive portions of a sine wave.

Solution

The mean square of a wave $x(t)$ is defined as

$$\overline{x^2} = \frac{\displaystyle\int x^2\,dt}{\displaystyle\int dt}.$$

Consider a sine wave with general amplitude A and period $2\pi/\omega$, $A\sin\omega t$. The positive portion of it occurs for the first half of the period, and the negative portion occurs for the second half.

$$x(t) = \begin{cases} A\sin\omega t & 0 < t < \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t < \frac{2\pi}{\omega} \end{cases}$$

Over one cycle, from 0 to $2\pi/\omega$, the mean square is

$$\begin{aligned}
\overline{x^2} &= \frac{\displaystyle\int_0^{\frac{\pi}{\omega}} (A\sin\omega t)^2\,dt + \int_{\frac{\pi}{\omega}}^{\frac{2\pi}{\omega}} (0)\,dt}{\displaystyle\int_0^{\frac{2\pi}{\omega}} dt} \\
&= \frac{\displaystyle\int_0^{\frac{\pi}{\omega}} A^2\sin^2\omega t\,dt}{\frac{2\pi}{\omega}} \\
&= \frac{\omega A^2}{2\pi}\int_0^{\frac{\pi}{\omega}} \sin^2\omega t\,dt \\
&= \frac{\omega A^2}{2\pi}\int_0^{\frac{\pi}{\omega}} \frac{1}{2}(1-\cos 2\omega t)\,dt \\
&= \frac{\omega A^2}{4\pi}\left(t - \frac{1}{2\omega}\sin 2\omega t\right)\Bigg|_0^{\frac{\pi}{\omega}} \\
&= \frac{\omega A^2}{4\pi}\left(\frac{\pi}{\omega}\right) \\
&= \frac{A^2}{4}.
\end{aligned}$$

Therefore, taking the square root of $\overline{x^2}$, the root mean square of the wave is

$$x_{\text{rms}} = \frac{A}{2}.$$