Problem 1.14

Determine the mean square value of the sawtooth wave of Prob. 1.12. Do this two ways, from the squared curve and from the Fourier series.

Solution

The mean square of a wave $x(\theta)$ is defined as

$$\overline{x^2} = \frac{\displaystyle\int x^2 \, d\theta}{\displaystyle\int d\theta}.$$

Below is the sawtooth wave of Prob. 1.12.

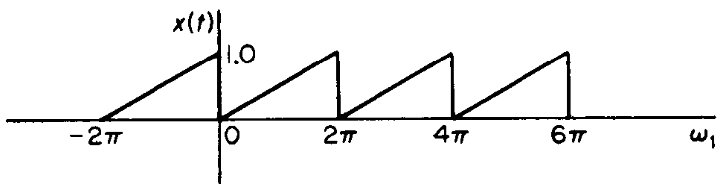


FIGURE P1.12.

It repeats itself every 2π radians and thus has period 2π.

From the Squared Curve

From 0 to 2π, one whole cycle, the wave is represented by $x(\theta) = (1/2\pi)\theta$. The mean square of the whole wave can be found by integrating over this cycle.

$$\begin{aligned}
\overline{x^2} &= \frac{\displaystyle\int_0^{2\pi} \left(\frac{1}{2\pi}\theta\right)^2 d\theta}{\displaystyle\int_0^{2\pi} d\theta} \\
&= \frac{\displaystyle\frac{1}{4\pi^2}\int_0^{2\pi} \theta^2 d\theta}{2\pi} \\
&= \frac{1}{8\pi^3} \cdot \frac{\theta^3}{3}\bigg|_0^{2\pi} \\
&= \frac{1}{8\pi^3} \cdot \frac{(2\pi)^3}{3} \\
&= \frac{1}{3}.
\end{aligned}$$

From the Fourier Series

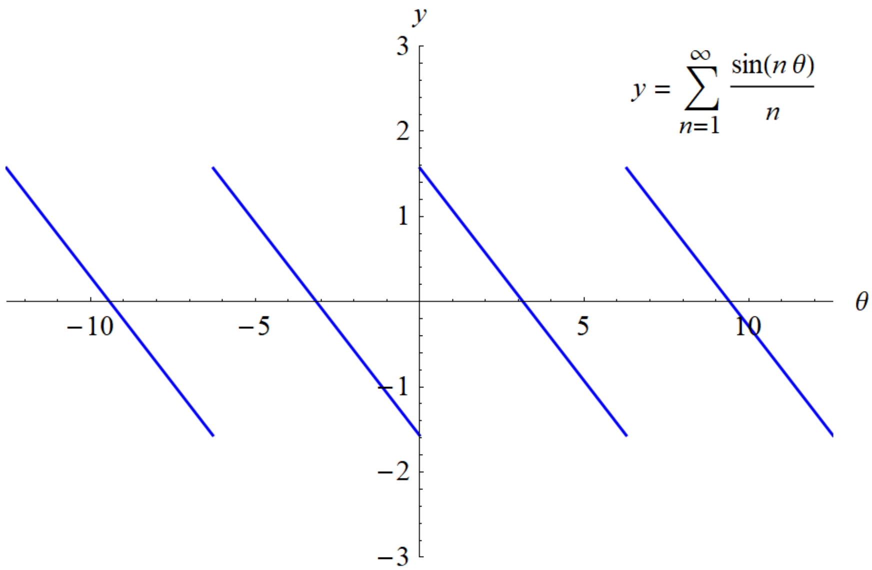
The Fourier series for the sawtooth wave was found in Problem 1.12.

$$x(\theta) = \frac{1}{2} - \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin n\theta}{n}$$

The infinite series can be summed by writing sine in terms of exponential functions and then recognizing the function associated with the Maclaurin series.

$$\begin{aligned}
\sum_{n=1}^{\infty}\frac{\sin n\theta}{n} &= \lim_{r\to 1}\sum_{n=1}^{\infty}\frac{r^n \sin n\theta}{n} \\
&= \lim_{r\to 1}\sum_{n=1}^{\infty}\frac{r^n}{n}\left(\frac{e^{in\theta}-e^{-in\theta}}{2i}\right) \\
&= \lim_{r\to 1}\left(\sum_{n=1}^{\infty}\frac{r^n e^{in\theta}}{2in} - \sum_{n=1}^{\infty}\frac{r^n e^{-in\theta}}{2in}\right) \\
&= \lim_{r\to 1}\frac{1}{2i}\left[\sum_{n=1}^{\infty}\frac{(re^{i\theta})^n}{n} - \sum_{n=1}^{\infty}\frac{(re^{-i\theta})^n}{n}\right] \\
&= \lim_{r\to 1}\frac{1}{2i}\left[-\ln(1-re^{i\theta}) + \ln(1-re^{-i\theta})\right] \\
&= \lim_{r\to 1}\frac{1}{2i}\ln\frac{1-re^{-i\theta}}{1-re^{i\theta}} \\
&= \frac{1}{2i}\ln\frac{1-e^{-i\theta}}{1-e^{i\theta}} \\
&= \frac{1}{2i}\ln\left(e^{-i\theta}\cdot\frac{e^{i\theta}-1}{1-e^{i\theta}}\right) \\
&= \frac{1}{2i}\ln(-e^{-i\theta}) \\
&= \frac{i}{2}\ln(-e^{i\theta}) \\
&= \frac{i}{2}\ln[e^{i(\pi+2\pi n)}e^{i\theta}], \quad n = 0, \pm 1, \pm 2, \ldots \\
&= \frac{i}{2}\ln e^{i[\theta+(1+2n)\pi]}, \quad n = 0, \pm 1, \pm 2, \ldots \\
&= \begin{cases} \vdots & \\ -\frac{1}{2}(\theta+3\pi) & -4\pi < \theta \le -2\pi \\ -\frac{1}{2}(\theta+\pi) & -2\pi < \theta \le 0 \\ -\frac{1}{2}(\theta-\pi) & 0 < \theta \le 2\pi \\ -\frac{1}{2}(\theta-3\pi) & 2\pi < \theta \le 4\pi \\ \vdots & \end{cases}
\end{aligned}$$

The infinite series is essentially a reverse sawtooth function as illustrated in the figure below.



Any of the entries for the infinite series can be used, provided that we integrate over the proper interval. In $-2\pi < \theta \leq 0$, for example,

$$\begin{aligned} x(\theta) &= \frac{1}{2} - \frac{1}{\pi}\left[-\frac{1}{2}(\theta + \pi)\right] \\ &= 1 + \frac{\theta}{2\pi}, \end{aligned}$$

and the mean square is the same as before.

$$\begin{aligned} \overline{x^2} &= \frac{\int_{-2\pi}^{0} \left(1 + \frac{\theta}{2\pi}\right)^2 d\theta}{\int_{-2\pi}^{0} d\theta} \\ &= \frac{1}{2\pi}\int_{-2\pi}^{0} \left(1 + \frac{\theta}{\pi} + \frac{\theta^2}{4\pi^2}\right) d\theta \\ &= \frac{1}{2\pi}\left(\theta + \frac{\theta^2}{2\pi} + \frac{\theta^3}{12\pi^2}\right)\Bigg|_{-2\pi}^{0} \\ &= \frac{1}{2\pi}\left[-(-2\pi) - \frac{(-2\pi)^2}{2\pi} - \frac{(-2\pi)^3}{12\pi^2}\right] \\ &= 1 - 1 + \frac{4\pi^2}{12\pi^2} \\ &= \frac{1}{3} \end{aligned}$$