Problem 1.4

Find the sum of two harmonic functions of equal amplitude but of slightly different frequencies. Discuss the beating phenomena that result from this sum.

Solution

Let the two harmonic functions, with amplitude A and linear frequencies f_1 and f_2, be denoted as $x_1(t)$ and $x_2(t)$, respectively.

$$x_1(t) = A\sin 2\pi f_1 t$$
$$x_2(t) = A\sin 2\pi f_2 t$$

Their sum is

$$\begin{aligned} x_1(t) + x_2(t) &= A\sin 2\pi f_1 t + A\sin 2\pi f_2 t \\ &= A(\sin 2\pi f_1 t + \sin 2\pi f_2 t). \end{aligned}$$

Use the sum-to-product formula for sines.

$$\begin{aligned} &= A\left[2\sin\left(\frac{2\pi f_1 t + 2\pi f_2 t}{2}\right)\cos\left(\frac{2\pi f_1 t - 2\pi f_2 t}{2}\right)\right] \\ &= 2A\cos\left(2\pi\frac{f_1 - f_2}{2}t\right)\sin\left(2\pi\frac{f_1 + f_2}{2}t\right) \\ &= a(t)\sin 2\pi\bar{f}t \end{aligned}$$

Thus, the sum of two harmonic functions is another harmonic function with time-dependent amplitude $a(t)$ and average frequency $\bar{f}$. This variation in amplitude results in the beat phenomenon. At the times when the two waves, $x_1(t)$ and $x_2(t)$, are in phase with each other, they interfere constructively, resulting in a maximum amplitude $2A$. On the other hand, when the two waves, $x_1(t)$ and $x_2(t)$, are out of phase with one another, they interfere destructively, resulting in a minimum amplitude 0. Figure 1 shows the behavior of the harmonic functions for the case that $f_1 = 10$ cycles per second and $f_2 = 9$ cycles per second.

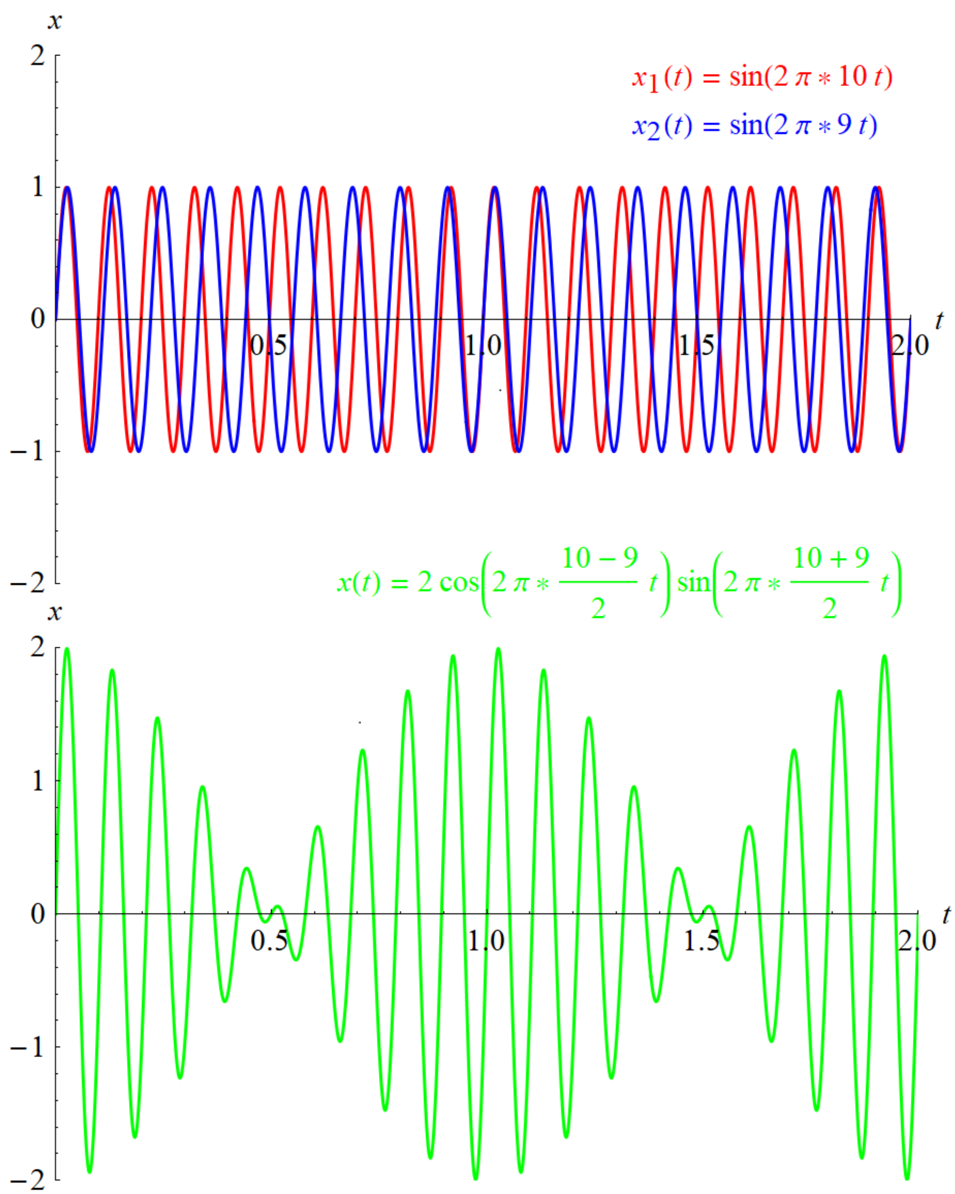

Figure 1: This figure shows the graphs of the two harmonic functions (in red and blue) versus time along with the graph of their sum (in green) versus time below. One beat occurs every second in this example, i.e. from $t = 0.5$ s to $t = 1.5$ s. Note that the beat frequency is $|f_1 - f_2|$ in general.