Problem 1.10

If the origin of the square wave of Prob. 1.9 is shifted to the right by $\pi/2$, determine the Fourier series.

Solution

The difference between this problem and the previous one is that the function f is now even.

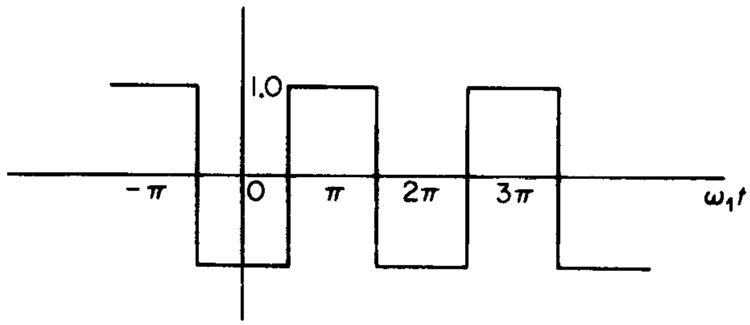

FIGURE P1.10.

Notice that the wave repeats itself every 2π radians. The general Fourier series for a 2π-periodic function is

$$f(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta. \tag{1}$$

In order to take advantage of the properties of even and odd functions, we aim to find the Fourier series by integrating over the symmetric interval $(-\pi, \pi)$. Integrate both sides of equation (1) with respect to θ from $-\pi$ to π to solve for A_0.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(\theta)\, d\theta &= \int_{-\pi}^{\pi} \left(A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \right) d\theta \\
&= \int_{-\pi}^{\pi} A_0 \, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} A_n \cos n\theta \, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} B_n \sin n\theta \, d\theta \\
&= A_0 \int_{-\pi}^{\pi} d\theta + \sum_{n=1}^{\infty} A_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta \, d\theta}_{=\,0} + \sum_{n=1}^{\infty} B_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta \, d\theta}_{=\,0} \\
&= 2\pi A_0
\end{aligned}$$

Consequently,

$$\begin{aligned}
A_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta \\
&= \frac{1}{2\pi} \left[\int_{-\pi}^{-\pi/2} (1)\, d\theta + \int_{-\pi/2}^{\pi/2} (-1)\, d\theta + \int_{\pi/2}^{\pi} (1)\, d\theta \right] \\
&= \frac{1}{2\pi} \left(\frac{\pi}{2} - \pi + \frac{\pi}{2} \right) \\
&= 0.
\end{aligned}$$

The fact that $A_0 = 0$ could have been predicted from the fact that the average of the wave is zero.

A_n will now be determined. Multiply both sides of equation (1) by $\cos m\theta$, where m is an integer,

$$f(\theta)\cos m\theta = A_0 \cos m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta$$

and then integrate both sides with respect to θ from $-\pi$ to π.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(\theta)\cos m\theta \, d\theta &= \int_{-\pi}^{\pi} \left(A_0 \cos m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta \right) d\theta \\
&= \int_{-\pi}^{\pi} A_0 \cos m\theta \, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta \, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta \, d\theta \\
&= A_0 \underbrace{\int_{-\pi}^{\pi} \cos m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta \cos m\theta \, d\theta}_{=\pi \text{ only if } n=m} + \sum_{n=1}^{\infty} B_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta \cos m\theta \, d\theta}_{=0 \text{ for all } n \text{ and } m}
\end{aligned}$$

Because the trigonometric functions are orthogonal, only one term in the first infinite series remains as a result of the integration. All other terms vanish.

$$\int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta = A_n(\pi)$$

Consequently,

$$\begin{aligned}
A_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta)\cos n\theta \, d\theta \\
&= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} (1)\cos n\theta \, d\theta + \int_{-\pi/2}^{\pi/2} (-1)\cos n\theta \, d\theta + \int_{\pi/2}^{\pi} (1)\cos n\theta \, d\theta \right] \\
&= \frac{1}{\pi} \left(-\frac{\sin \frac{n\pi}{2}}{n} - \frac{2\sin \frac{n\pi}{2}}{n} - \frac{\sin \frac{n\pi}{2}}{n} \right) \\
&= -\frac{4}{n\pi} \sin \frac{n\pi}{2}.
\end{aligned}$$

B_n will now be determined. Multiply both sides of equation (1) by $\sin m\theta$, where m is an integer,

$$f(\theta)\sin m\theta = A_0 \sin m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta$$

and then integrate both sides with respect to θ from $-\pi$ to π.

$$\begin{aligned}
\int_{-\pi}^{\pi} f(\theta)\sin m\theta\, d\theta &= \int_{-\pi}^{\pi} \left(A_0 \sin m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta \right) d\theta \\
&= \int_{-\pi}^{\pi} A_0 \sin m\theta\, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta\, d\theta + \int_{-\pi}^{\pi} \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta\, d\theta \\
&= A_0 \underbrace{\int_{-\pi}^{\pi} \sin m\theta\, d\theta}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_{-\pi}^{\pi} \cos n\theta \sin m\theta\, d\theta}_{=0 \text{ for all } n \text{ and } m} + \sum_{n=1}^{\infty} B_n \underbrace{\int_{-\pi}^{\pi} \sin n\theta \sin m\theta\, d\theta}_{=\pi \text{ only if } n=m}
\end{aligned}$$

Because the trigonometric functions are orthogonal, only one term in the second infinite series remains as a result of the integration. All other terms vanish.

$$\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta = B_n(\pi)$$

Consequently,

$$\begin{aligned}
B_n &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin n\theta\, d\theta \\
&= \frac{1}{\pi} \left[\int_{-\pi}^{-\pi/2} (1) \sin n\theta\, d\theta + \int_{-\pi/2}^{\pi/2} (-1) \sin n\theta\, d\theta + \int_{\pi/2}^{\pi} (1) \sin n\theta\, d\theta \right] \\
&= \frac{1}{\pi} \left(\frac{-\cos\frac{n\pi}{2} + \cos n\pi}{n} + 0 + \frac{\cos\frac{n\pi}{2} - \cos n\pi}{n} \right) \\
&= 0.
\end{aligned}$$

The fact that $B_n = 0$ could have been predicted from the fact that the wave is an even function. With the coefficients determined, equation (1) becomes

$$f(\theta) = \sum_{n=1}^{\infty} \frac{-4}{n\pi} \sin \frac{n\pi}{2} \cos n\theta.$$

If n is even, the coefficient vanishes, so the series can be simplified (that is, made to converge faster) by summing over the odd integers only. Let $n = 2k - 1$ in the sum.

$$f(\theta) = \sum_{2k-1=1}^{\infty} \frac{-4}{(2k-1)\pi} \sin \frac{(2k-1)\pi}{2} \cos[(2k-1)\theta] = \sum_{k=1}^{\infty} \frac{-4}{(2k-1)\pi} [-(-1)^k] \cos[(2k-1)\theta]$$

Therefore, replacing θ with $\omega_1 t$,

$$f(t) = \frac{4}{\pi} \sum_{k=1}^{\infty} \frac{(-1)^k}{2k-1} \cos[(2k-1)\omega_1 t].$$

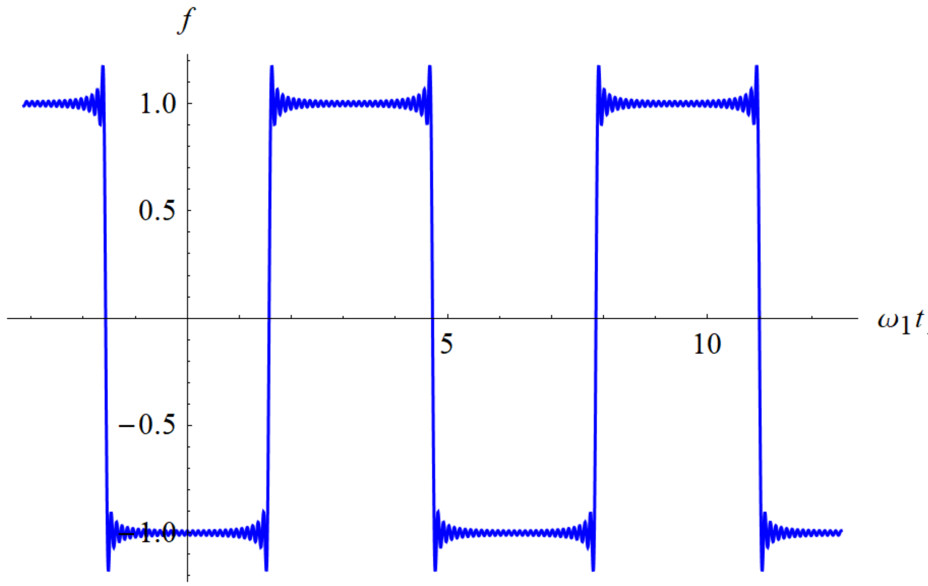

Figure 1: This figure shows a plot of $f(t)$ versus t using only the first 30 terms in the infinite series. The more terms that are used in the series, the more it looks like the function in Figure P1.10.