Problem 1.16

Determine the Fourier series of a series of rectangular pulses shown in Fig. P1.16. Plot c_n and ϕ_n versus n when $k = \frac{2}{3}$.

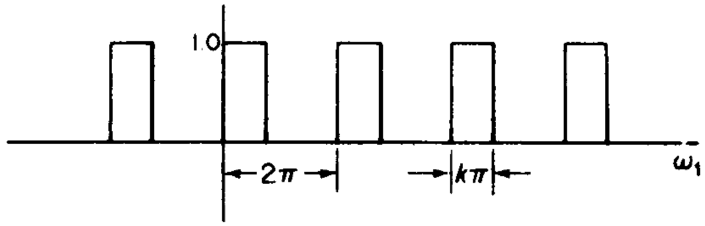


FIGURE P1.16.

Solution

Notice that the wave repeats itself every 2π radians. The general Fourier series for a 2π-periodic function is

$$x(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta. \tag{1}$$

Integrate both sides of equation (1) with respect to θ from 0 to 2π to solve for A_0.

$$\begin{aligned}
\int_0^{2\pi} x(\theta)\,d\theta &= \int_0^{2\pi} \left(A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta\right) d\theta \\
&= \int_0^{2\pi} A_0\,d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} A_n \cos n\theta\,d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} B_n \sin n\theta\,d\theta \\
&= A_0 \int_0^{2\pi} d\theta + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{2\pi} \cos n\theta\,d\theta}_{=0} + \sum_{n=1}^{\infty} B_n \underbrace{\int_0^{2\pi} \sin n\theta\,d\theta}_{=0} \\
&= 2\pi A_0
\end{aligned}$$

Consequently,

$$\begin{aligned}
A_0 &= \frac{1}{2\pi} \int_0^{2\pi} x(\theta)\,d\theta \\
&= \frac{1}{2\pi} \left[\int_0^{k\pi} (1)\,d\theta + \int_{k\pi}^{2\pi} (0)\,d\theta\right] \\
&= \frac{1}{2\pi}(k\pi) \\
&= \frac{k}{2}.
\end{aligned}$$

A_n will now be determined. Multiply both sides of equation (1) by $\cos m\theta$, where m is an integer,

$$x(\theta)\cos m\theta = A_0 \cos m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta$$

and then integrate both sides with respect to θ from 0 to 2π.

$$\begin{aligned}
\int_0^{2\pi} x(\theta)\cos m\theta \, d\theta &= \int_0^{2\pi} \left(A_0 \cos m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta \right) d\theta \\
&= \int_0^{2\pi} A_0 \cos m\theta \, d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} A_n \cos n\theta \cos m\theta \, d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} B_n \sin n\theta \cos m\theta \, d\theta \\
&= A_0 \underbrace{\int_0^{2\pi} \cos m\theta \, d\theta}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{2\pi} \cos n\theta \cos m\theta \, d\theta}_{=\pi \text{ only if } n=m} + \sum_{n=1}^{\infty} B_n \underbrace{\int_0^{2\pi} \sin n\theta \cos m\theta \, d\theta}_{=0 \text{ for all } n \text{ and } m}
\end{aligned}$$

Because the trigonometric functions are orthogonal, only one term in the first infinite series remains as a result of the integration. All other terms vanish.

$$\int_0^{2\pi} x(\theta)\cos n\theta \, d\theta = A_n(\pi)$$

Consequently,

$$\begin{aligned}
A_n &= \frac{1}{\pi} \int_0^{2\pi} x(\theta)\cos n\theta \, d\theta \\
&= \frac{1}{\pi} \left[\int_0^{k\pi} (1)\cos n\theta \, d\theta + \int_{k\pi}^{2\pi} (0)\cos n\theta \, d\theta \right] \\
&= \frac{1}{\pi} \left(\frac{\sin kn\pi}{n} \right) \\
&= \frac{\sin kn\pi}{n\pi}.
\end{aligned}$$

B_n will now be determined. Multiply both sides of equation (1) by $\sin m\theta$, where m is an integer,

$$x(\theta)\sin m\theta = A_0 \sin m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta$$

and then integrate both sides with respect to θ from 0 to 2π.

$$\begin{aligned}
\int_0^{2\pi} x(\theta)\sin m\theta\, d\theta &= \int_0^{2\pi} \left(A_0 \sin m\theta + \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta \right) d\theta \\
&= \int_0^{2\pi} A_0 \sin m\theta\, d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} A_n \cos n\theta \sin m\theta\, d\theta + \int_0^{2\pi} \sum_{n=1}^{\infty} B_n \sin n\theta \sin m\theta\, d\theta \\
&= A_0 \underbrace{\int_0^{2\pi} \sin m\theta\, d\theta}_{=0} + \sum_{n=1}^{\infty} A_n \underbrace{\int_0^{2\pi} \cos n\theta \sin m\theta\, d\theta}_{=0 \text{ for all } n \text{ and } m} + \sum_{n=1}^{\infty} B_n \underbrace{\int_0^{2\pi} \sin n\theta \sin m\theta\, d\theta}_{=\pi \text{ only if } n=m}
\end{aligned}$$

Because the trigonometric functions are orthogonal, only one term in the second infinite series remains as a result of the integration. All other terms vanish.

$$\int_0^{2\pi} x(\theta)\sin n\theta\, d\theta = B_n(\pi)$$

Consequently,

$$\begin{aligned}
B_n &= \frac{1}{\pi}\int_0^{2\pi} x(\theta)\sin n\theta\, d\theta \\
&= \frac{1}{\pi}\left[\int_0^{k\pi} (1)\sin n\theta\, d\theta + \int_{k\pi}^{2\pi} (0)\sin n\theta\, d\theta\right] \\
&= \frac{1}{\pi}\left(\frac{1-\cos kn\pi}{n}\right) \\
&= \frac{1-\cos kn\pi}{n\pi}.
\end{aligned}$$

The Fourier series for the series of rectangular pulses is then

$$\begin{aligned}
x(\theta) &= A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + \sum_{n=1}^{\infty} B_n \sin n\theta \\
&= \frac{k}{2} + \sum_{n=1}^{\infty} \frac{\sin kn\pi}{n\pi}\cos n\theta + \sum_{n=1}^{\infty} \frac{1-\cos kn\pi}{n\pi}\sin n\theta \\
&= \frac{k}{2} + \sum_{n=1}^{\infty} \frac{\sin kn\pi \cos n\theta - \cos kn\pi \sin n\theta + \sin n\theta}{n\pi} \\
&= \frac{k}{2} + \sum_{n=1}^{\infty} \frac{\sin(kn\pi - n\theta) + \sin n\theta}{n\pi} \\
&= \frac{k}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\sin[n(k\pi - \theta)] + \sin n\theta}{n}.
\end{aligned}$$

Therefore, replacing θ with $\omega_1 t$,

$$\boxed{x(t) = \frac{k}{2} + \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin[n(k\pi - \omega_1 t)] + \sin n\omega_1 t}{n}.}$$

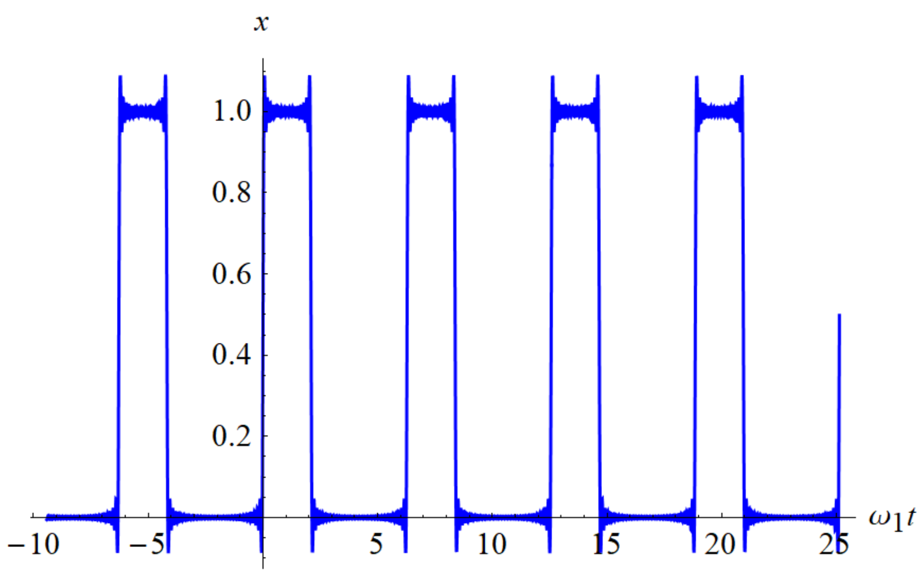
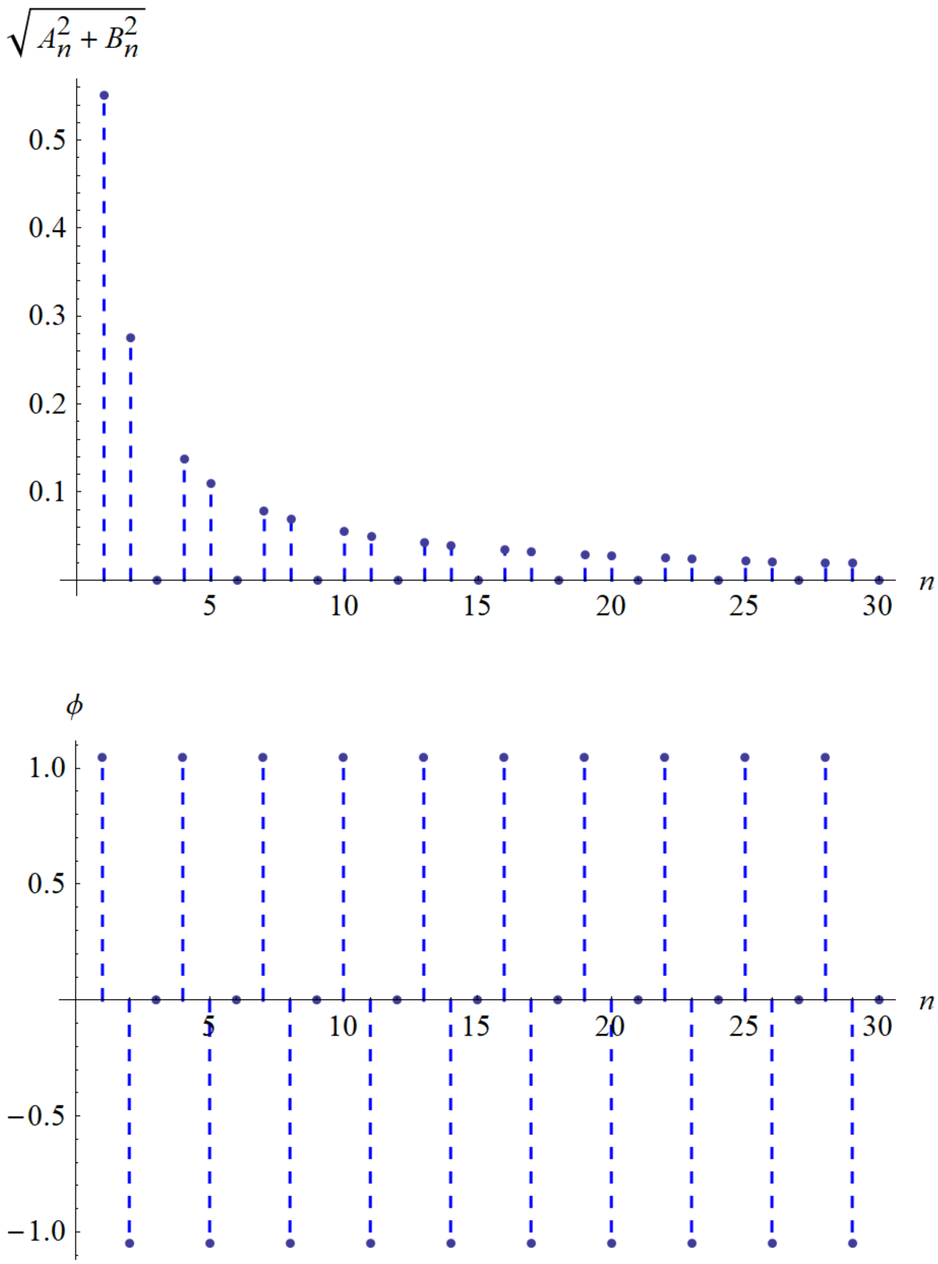


Figure 1: This figure shows a plot of $x(t)$ versus $\omega_1 t$ for $k = \frac{2}{3}$ using only the first 50 terms in the infinite series.

Compute the quantities, $|2c_n|$ and ϕ.

$$\begin{aligned}
|2c_n| &= \sqrt{A_n^2 + B_n^2} \\
&= \sqrt{\frac{\sin^2 kn\pi}{n^2\pi^2} + \frac{(1 - \cos kn\pi)^2}{n^2\pi^2}} \\
&= \sqrt{\frac{\sin^2 kn\pi + 1 - 2\cos kn\pi + \cos^2 kn\pi}{n^2\pi^2}} \\
&= \sqrt{\frac{2 - 2\cos kn\pi}{n^2\pi^2}} \\
&= \frac{\sqrt{2(1 - \cos kn\pi)}}{n\pi} \\
\phi &= \tan^{-1}\frac{B_n}{A_n} \\
&= \tan^{-1}\left(\frac{1 - \cos kn\pi}{\sin kn\pi}\right) \\
&= \tan^{-1}\tan\frac{kn\pi}{2}
\end{aligned}$$

The Fourier spectrum consists of two plots, $\sqrt{A_n^2 + B_n^2}$ versus n and ϕ versus n.


$\sqrt{A_n^2 + B_n^2}$
0.5
0.4
0.3
0.2
0.1
5
10
15
20
25
30
n
ϕ
1.0
0.5
−0.5
−1.0
5
10
15
20
25
30
n