Problem 1.17

Write the equation for the displacement s of the piston in the crank-piston mechanism shown in Fig. P1.17, and determine the harmonic components and their relative magnitudes. If $r/l = \frac{1}{3}$, what is the ratio of the second harmonic compared to the first?

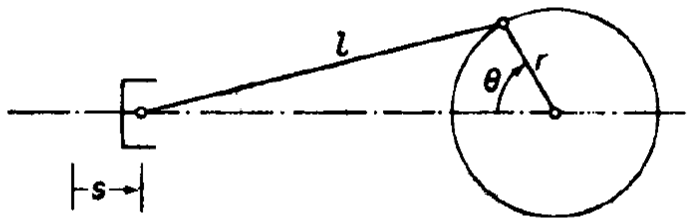


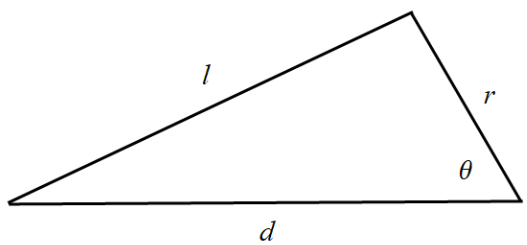
FIGURE P1.17.

Solution

Notice that if $\theta = 0$, then the crank lies completely horizontal, making the displacement of the piston $s = 0$. s can be calculated by

$$s = r + l - d,$$

where d is the length of the longest leg in the triangle below.



Use the law of cosines to obtain an equation for d.

$$l^2 = r^2 + d^2 - 2rd\cos\theta$$
$$d^2 - (2r\cos\theta)d + (r^2 - l^2) = 0$$

Use the quadratic formula to solve for d.

$$\begin{aligned} d &= \frac{2r\cos\theta \pm \sqrt{4r^2\cos^2\theta - 4(r^2 - l^2)}}{2} \\ &= r\cos\theta \pm \sqrt{r^2\cos^2\theta - (r^2 - l^2)} \\ &= r\cos\theta \pm \sqrt{r^2\cos^2\theta - r^2 + l^2} \\ &= r\cos\theta \pm \sqrt{l^2 - r^2(1 - \cos^2\theta)} \\ &= r\cos\theta \pm \sqrt{l^2 - r^2\sin^2\theta} \end{aligned}$$

We choose the solution with the plus sign because we want $s = 0$ when $\theta = 0$.

$$d = r\cos\theta + \sqrt{l^2 - r^2\sin^2\theta}$$

Now that d is known, we can determine s.

$$\begin{aligned} s &= r + l - d \\ &= r + l - r\cos\theta - \sqrt{l^2 - r^2\sin^2\theta} \\ &= r - r\cos\theta + l - l\sqrt{1 - \frac{r^2}{l^2}\sin^2\theta} \end{aligned}$$

Therefore,

$$\boxed{s = r(1-\cos\theta) + l\left(1 - \sqrt{1 - \frac{r^2}{l^2}\sin^2\theta}\right).}$$

s has two harmonic components, s_1 and s_2.

$$\begin{aligned} s_1 &= r(1-\cos\theta) \\ s_2 &= l\left(1 - \sqrt{1 - \frac{r^2}{l^2}\sin^2\theta}\right) \end{aligned}$$

The magnitude of s_2 relative to s_1 is

$$\begin{aligned} \frac{s_2}{s_1} &= \frac{l\left(1 - \sqrt{1 - \frac{r^2}{l^2}\sin^2\theta}\right)}{r(1-\cos\theta)} \\ &= \frac{l}{r}\frac{1 - \sqrt{1 - \frac{r^2}{l^2}\sin^2\theta}}{1-\cos\theta}. \end{aligned}$$

If $r/l = 1/3$, then the ratio of s_2 to s_1 is

$$\begin{aligned} &= 3\frac{1 - \sqrt{1 - \frac{1}{9}\sin^2\theta}}{1-\cos\theta} \\ &= \frac{3 - 3\sqrt{1 - \frac{1}{9}\sin^2\theta}}{1-\cos\theta} \\ &= \frac{3 - \sqrt{9 - \sin^2\theta}}{1-\cos\theta}. \end{aligned}$$