Problem 1.19

Determine the mean square value of the triangular wave of Fig. P1.11.

Solution

The mean square of a wave $x(\theta)$ is defined as

$$\overline{x^2} = \frac{\int x^2 \, d\theta}{\int d\theta}.$$

Below is the triangular wave of Fig. P1.11.

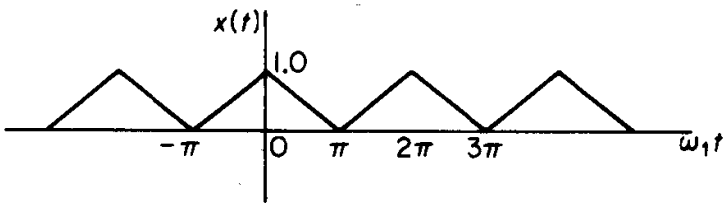

FIGURE P1.11.

It repeats itself every 2π radians and thus has period 2π. Its first cycle can be represented piecewise as

$$x(\theta) = \begin{cases} \frac{1}{\pi}\theta + 1 & -\pi < \theta < 0 \\ -\frac{1}{\pi}\theta + 1 & 0 < \theta < \pi \end{cases}.$$

The mean square of the whole wave can be found by integrating over one cycle.

$$\begin{aligned}
\overline{x^2} &= \frac{\displaystyle\int_{-\pi}^{0} \left(\frac{1}{\pi}\theta + 1\right)^2 d\theta + \int_{0}^{\pi} \left(-\frac{1}{\pi}\theta + 1\right)^2 d\theta}{\displaystyle\int_{-\pi}^{\pi} d\theta} \\
&= \frac{\displaystyle\int_{-\pi}^{0} \left(\frac{1}{\pi^2}\theta^2 + \frac{2}{\pi}\theta + 1\right) d\theta + \int_{0}^{\pi} \left(\frac{1}{\pi^2}\theta^2 - \frac{2}{\pi}\theta + 1\right) d\theta}{2\pi} \\
&= \frac{\left(\frac{1}{3\pi^2}\theta^3 + \frac{1}{\pi}\theta^2 + \theta\right)\Big|_{-\pi}^{0} + \left(\frac{1}{3\pi^2}\theta^3 - \frac{1}{\pi}\theta^2 + \theta\right)\Big|_{0}^{\pi}}{2\pi} \\
&= \frac{-\frac{1}{3\pi^2}(-\pi)^3 - \frac{1}{\pi}(-\pi)^2 - (-\pi) + \frac{1}{3\pi^2}(\pi)^3 - \frac{1}{\pi}(\pi)^2 + (\pi)}{2\pi} \\
&= \frac{\frac{2}{3}\pi - 2\pi + 2\pi}{2\pi}
\end{aligned}$$

Therefore, the mean square of the triangular wave is

$$\overline{x^2} = \frac{1}{3}.$$