Problem 2.5

A mass m_1 hangs from a spring k N/m and is in static equilibrium. A second mass m_2 drops through a height h and sticks to m_1 without rebound, as shown in Fig. P2.5. Determine the subsequent motion.

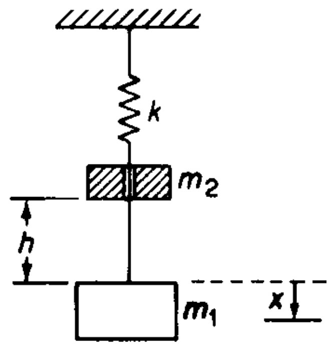


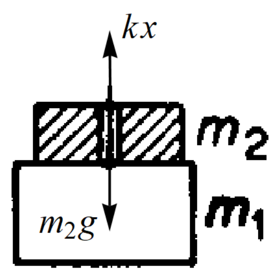
FIGURE P2.5.

Solution

The equation of motion is obtained by using Newton's second law.

$$\sum \mathbf{F} = m\mathbf{a}$$

The subsequent motion is due to the collision of m_2 with m_1 and is influenced by two forces, namely the gravitational force of m_2 and the spring force. These are illustrated in the free-body diagram of m_1 below.



Apply Newton's second law, taking the positive x-direction to be downward.

$$m_2g - kx = (m_1 + m_2)\ddot{x}$$

The inertial force is $(m_1 + m_2)\ddot{x}$ because the bodies move together as one. The differential equation governing the subsequent motion is thus

$$(m_1 + m_2)\ddot{x} + kx = m_2g. \tag{1}$$

Since this is a second-order linear inhomogeneous equation, the general solution is the sum of a complementary solution and a particular solution.

$$x = x_c + x_p$$

The complementary solution satisfies the associated homogeneous equation.

$$(m_1 + m_2)\ddot{x}_c + kx_c = 0$$

Bring kx_c to the right side and divide both sides by $m_1 + m_2$.

$$\ddot{x}_c = -\frac{k}{m_1 + m_2}x_c$$

Its solution can be written in terms of sine and cosine.

$$x_c(t) = A\cos\left(\sqrt{\frac{k}{m_1 + m_2}}t\right) + B\sin\left(\sqrt{\frac{k}{m_1 + m_2}}t\right)$$

Because the inhomogeneous term in equation (1) is constant in time, the particular solution is constant in time as well.

$$\underbrace{(m_1 + m_2)\ddot{x}_p}_{=0} + kx_p = m_2 g$$

The particular solution is then

$$x_p = \frac{m_2 g}{k},$$

which means the general solution to equation (1) is

$$x(t) = A\cos\left(\sqrt{\frac{k}{m_1 + m_2}}t\right) + B\sin\left(\sqrt{\frac{k}{m_1 + m_2}}t\right) + \frac{m_2 g}{k}.$$

In order to determine the arbitrary constants, A and B, we need two initial conditions, one for x and one for $\dot{x}$. The initial position is $x(0) = 0$ from the selected coordinate system. Let v_1 represent the speed of m_2 the instant before it collides with m_1, and let v_2 represent the speed of $(m_1 + m_2)$ the instant after the collision. There are no dissipative forces involved, so the total mechanical energy is conserved: as m_2 falls a distance h, the potential energy is converted to kinetic energy.

$$m_2 gh = \frac{1}{2}m_2 v_1^2$$

Solve this equation for v_1.

$$v_1 = \sqrt{2gh}$$

Assuming the collision force is much greater than the external forces, they can be neglected. The total momentum is conserved as a consequence, meaning that the momentum is the same before and after the collision.

$$m_2 v_1 = (m_1 + m_2)v_2$$

Solve this equation for v_2 and substitute the result for v_1.

$$\begin{aligned} v_2 &= \frac{m_2}{m_1 + m_2}v_1 \\ &= \frac{m_2}{m_1 + m_2}\sqrt{2gh} \end{aligned}$$

This is the second initial condition.

$$\dot{x}(0) = \frac{m_2}{m_1 + m_2}\sqrt{2gh}$$

Apply both initial conditions now to determine A and B.

$$x(0) = A + \frac{m_2 g}{k} = 0 \qquad \rightarrow \qquad A = -\frac{m_2 g}{k}$$

$$\dot{x}(0) = B\sqrt{\frac{k}{m_1 + m_2}} = \frac{m_2}{m_1 + m_2}\sqrt{2gh} \qquad \rightarrow \qquad B = m_2\sqrt{\frac{2gh}{k(m_1 + m_2)}}$$

As a result, the general solution to equation (1) becomes

$$x(t) = -\frac{m_2 g}{k}\cos\left(\sqrt{\frac{k}{m_1 + m_2}}t\right) + m_2\sqrt{\frac{2gh}{k(m_1 + m_2)}}\sin\left(\sqrt{\frac{k}{m_1 + m_2}}t\right) + \frac{m_2 g}{k}.$$

Therefore, the subsequent motion is

$$x(t) = \frac{m_2 g}{k}\left[1 - \cos\left(\sqrt{\frac{k}{m_1 + m_2}}t\right)\right] + m_2\sqrt{\frac{2gh}{k(m_1 + m_2)}}\sin\left(\sqrt{\frac{k}{m_1 + m_2}}t\right).$$