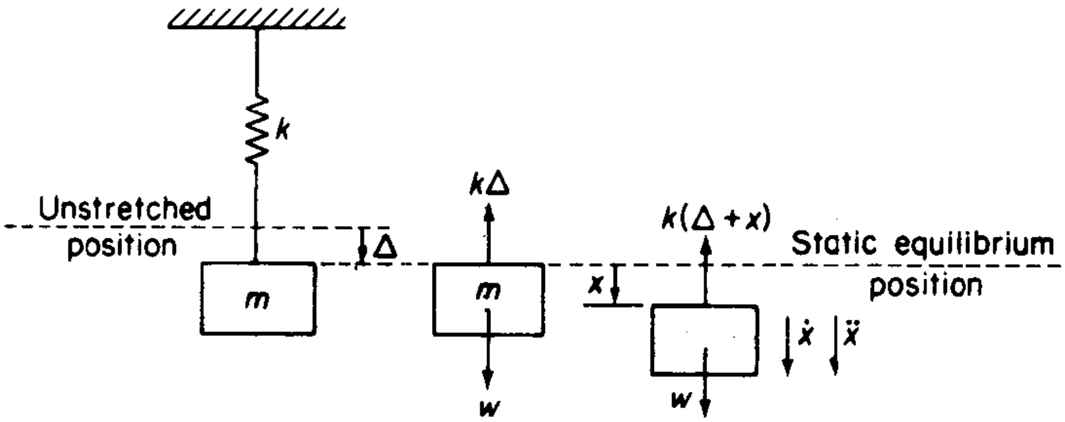
Problem 2.6

The ratio k/m of a spring-mass system is given as 4.0. If the mass is deflected 2 cm down, measured from its equilibrium position, and given an upward velocity of 8 cm/s, determine its amplitude and maximum acceleration.

Solution

The figure below illustrates the spring-mass system, the free-body diagram at equilibrium, and the free-body diagram at nonequilibrium.



The equation of motion for the spring-mass system is obtained by using Newton's second law twice, once at equilibrium and once at nonequilibrium.

$$\sum \mathbf{F} = m\mathbf{a}$$

Take the positive x direction to be downward as indicated in the figure.

At equilibrium	At nonequilibrium
$w - k\Delta = m(0)$	$w - k(\Delta + x) = m\ddot{x}$
$w - k\Delta = 0$	$w - k\Delta - kx = m\ddot{x}$
	$-kx = m\ddot{x}$
	$\ddot{x} = -\frac{k}{m}x$

The differential equation to solve for the motion of the spring-mass system is thus

$$\ddot{x} = -4x.$$

Its general solution can be written as

$$x(t) = A\cos(2t + \phi),$$

where A and ϕ are the amplitude and phase of the motion, respectively. Apply the initial conditions,

$$x(0) = 2\,\cancel{\text{cm}} \times \frac{1\text{ m}}{100\,\cancel{\text{cm}}} = 0.02\text{ m}$$

$$\dot{x}(0) = -8\,\frac{\cancel{\text{cm}}}{\text{s}} \times \frac{1\text{ m}}{100\,\cancel{\text{cm}}} = -0.08\,\frac{\text{m}}{\text{s}},$$

in order to obtain a system of equations for them.

$$x(0) = A\cos\phi = 0.02 \tag{1}$$
$$\dot{x}(0) = 2A\sin\phi = -0.08 \quad \rightarrow \quad A\sin\phi = -0.04 \tag{2}$$

Square both sides of each equation and add them together to solve for A.

$$\begin{aligned} A^2\cos^2\phi + A^2\sin^2\phi &= 0.02^2 + (-0.04)^2 \\ A^2 &= 0.002 \\ A &= \sqrt{0.002} \\ &\approx 0.0447 \end{aligned}$$

Divide both sides of equation (2) by those of equation (1) to solve for ϕ.

$$\begin{aligned} \frac{A\sin\phi}{A\cos\phi} &= \frac{-0.04}{0.02} \\ \tan\phi &= -2 \\ \phi &= \tan^{-1}(-2) \\ &= -\tan^{-1}2 \\ &\approx -1.11 \end{aligned}$$

Thus, the position of the mass (in meters) is

$$\begin{aligned} x(t) &= \sqrt{0.002}\cos(2t - \tan^{-1}2) \\ &\approx 0.0447\cos(2t - 1.11). \end{aligned}$$

Take a derivative of $x(t)$ to find the velocity (in meters/second).

$$\begin{aligned} \dot{x}(t) &= -2\sqrt{0.002}\sin(2t - \tan^{-1}2) \\ &\approx -0.0894\sin(2t - 1.11) \end{aligned}$$

Take a derivative of $\dot{x}(t)$ to find the acceleration (in meters/second^2).

$$\begin{aligned} \ddot{x}(t) &= -4\sqrt{0.002}\cos(2t - \tan^{-1}2) \\ &\approx -0.179\cos(2t - 1.11) \end{aligned}$$

Therefore, the amplitude of the motion is about 0.0447 meters, and the maximum acceleration is about 0.179 meters/second^2.