

Exercise 12

Find the closed form function for the following Taylor series:

$$f(x) = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots$$

Solution

Looking in Appendix C.4, which lists Taylor series expansions for hyperbolic trigonometric functions, we see that the function is

$$f(x) = \tanh x.$$

We can show that it is by using long division.

$$\begin{aligned} \tanh x &= \frac{\sinh x}{\cosh x} = \frac{x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{1}{7!}x^7 + \dots}{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{1}{6!}x^6 + \dots} \\ &= \left(1 + \frac{1}{2}x^2 + \frac{1}{24}x^4 + \frac{1}{720}x^6 \right) \left(\frac{x - \frac{1}{3}x^3 + \frac{2}{15}x^5 - \frac{17}{315}x^7 + \dots}{x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \dots} \right) \\ &= \frac{x + \frac{1}{2}x^3 + \frac{1}{24}x^5 + \frac{1}{720}x^7 + \dots}{x + \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{5040}x^7 + \dots} \\ &= \frac{-\frac{1}{3}x^3 - \frac{1}{30}x^5 - \frac{1}{840}x^7 - \dots}{-\frac{1}{3}x^3 - \frac{1}{6}x^5 - \frac{1}{72}x^7 - \dots} \\ &= \frac{+\frac{2}{15}x^5 + \frac{4}{315}x^7 + \dots}{+\frac{2}{15}x^5 + \frac{1}{15}x^7 + \dots} \\ &= \frac{-\frac{17}{315}x^7 - \dots}{-\frac{17}{315}x^7 - \dots} \\ &= \dots \end{aligned}$$