Exercise 10

Find the general solution for the following initial value problems:

$$u'' + 9u = 0, \quad u(0) = 1,\ u'(0) = 0$$

Solution

This is a homogeneous linear ODE with constant coefficients, so the solution is of the form, $u = e^{rx}$.

$$u = e^{rx} \quad \rightarrow \quad u' = re^{rx} \quad \rightarrow \quad u'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} + 9e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 + 9 = 0$$

Factor the left side.

$$(r + 3i)(r - 3i) = 0$$

$r = -3i$ or $r = 3i$, so the general solution is

$$u(x) = C_1 e^{-3ix} + C_2 e^{3ix}.$$

But this can be written in terms of sine and cosine by using Euler's formula. Therefore,

$$u(x) = A\cos 3x + B\sin 3x.$$

Because we're given initial conditions, we can determine A and B.

$$u'(x) = -3A\sin 3x + 3B\cos 3x$$

$$\begin{aligned} u(0) &= A & \rightarrow \quad A = 1 \\ u'(0) &= 3B & \rightarrow \quad B = 0 \end{aligned}$$

Therefore,

$$u(x) = \cos 3x.$$

We can check that this is the solution. The first and second derivatives are

$$\begin{aligned} u' &= -3\sin 3x \\ u'' &= -9\cos 3x. \end{aligned}$$

Hence,

$$u'' + 9u = -\cancel{9\cos 3x} + \cancel{9\cos 3x} = 0,$$

which means this is the correct solution.