Exercise 11

Find the general solution for the following initial value problems:

$$u'' - 9u' = 0, \quad u(0) = 3,\ u'(0) = 9$$

Solution

This is a homogeneous linear ODE with constant coefficients, so the solution is of the form, $u = e^{rx}$.

$$u = e^{rx} \quad \to \quad u' = re^{rx} \quad \to \quad u'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - 9re^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - 9r = 0$$

Factor the left side.

$$r(r - 9) = 0$$

$r = 0$ or $r = 9$, so the general solution is

$$u(x) = C_1 e^{0x} + C_2 e^{9x} = C_1 + C_2 e^{9x}.$$

Because we have two initial conditions, we can determine C_1 and C_2.

$$u'(x) = 9C_2 e^{9x}$$

$$u(0) = C_1 + C_2 = 3$$
$$u'(0) = 9C_2 = 9$$

Solving this system of equations gives $C_1 = 2$ and $C_2 = 1$. Therefore,

$$u(x) = 2 + e^{9x}.$$

We can check that this is the solution. The first and second derivatives are

$$u' = 9e^{9x}$$
$$u'' = 81e^{9x}.$$

Hence,

$$u'' - 9u' = \cancel{81e^{9x}} - \cancel{9(9e^{9x})} = 0,$$

which means this is the correct solution.