# Exercise 20

Use the method of undetermined coefficients to find the particular solution for the following initial value problems:

$$u'' - 5u' + 4u = -1 + 4x, \quad u(0) = 3,\ u'(0) = 9$$

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**Solution**

This is an inhomogeneous ODE, so the general solution is the sum of the complementary and particular solutions.

$$u = u_c + u_p$$

The complementary solution is the solution to the associated homogeneous equation,

$$u_c'' - 5u_c' + 4u_c = 0.$$

This is a linear ODE with constant coefficients, so the solution will be of the form $u_c = e^{rx}$.

$$u_c = e^{rx} \quad \rightarrow \quad u_c' = re^{rx} \quad \rightarrow \quad u_c'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - 5re^{rx} + 4e^{rx} = 0.$$

Divide both sides by $e^{rx}$.

$$r^2 - 5r + 4 = 0$$

Factor the left side.

$$(r-1)(r-4) = 0$$

$r = 1$ or $r = 4$, so the complementary solution is

$$u_c(x) = C_1 e^x + C_2 e^{4x}.$$

Now we turn our attention to the particular solution. Because the inhomogeneous term is $-1+4x$, try a particular solution of the form, $u_p = A + Bx$. Plugging this form into the ODE yields

$$u_p'' - 5u_p' + 4u_p = -5B + 4(A + Bx) = (-5B + 4A) + 4Bx = -1 + 4x.$$

Now we match the coefficients to determine $A$ and $B$.

$$\begin{aligned} -5B + 4A &= -1 \\ 4B &= 4 \end{aligned}$$

The solution to this system of equations is $A = 1$ and $B = 1$. Thus, $u_p = 1 + x$. Therefore, the general solution to the ODE is

$$u(x) = C_1 e^x + C_2 e^{4x} + x + 1.$$

$C_1$ and $C_2$ can be determined since initial conditions are given.

$$u'(x) = C_1 e^x + 4C_2 e^{4x} + 1$$

$$u(0) = C_1 + C_2 + 1 = 3$$
$$u'(0) = C_1 + 4C_2 + 1 = 9$$

The solution to this system of equations is $C_1 = 0$ and $C_2 = 2$. Therefore,

$$u(x) = 2e^{4x} + x + 1.$$

We can check that this is the solution. The first and second derivatives are

$$u' = 8e^{4x} + 1$$
$$u'' = 32e^{4x}.$$

Hence,

$$u'' - 5u' + 4u = \cancel{32e^{4x}} - 5(\cancel{8e^{4x}} + 1) + 4(\cancel{2e^{4x}} + x + 1) = -1 + 4x,$$

which means this is the correct solution.