Exercise 6

Find the general solution for the following second order ODEs:

$$u'' + 4u = 0$$

Solution

This is a homogeneous linear ODE with constant coefficients, so the solution is of the form, $u = e^{rx}$.

$$u = e^{rx} \quad \rightarrow \quad u' = re^{rx} \quad \rightarrow \quad u'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} + 4e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 + 4 = 0$$

Factor the left side.

$$(r + 2i)(r - 2i) = 0$$

$r = -2i$ or $r = 2i$, so the general solution is

$$u(x) = C_1 e^{-2ix} + C_2 e^{2ix}.$$

But this can be written in terms of sine and cosine by using Euler's formula. Therefore,

$$u(x) = A\cos 2x + B\sin 2x.$$

We can check that this is the solution. The first and second derivatives are

$$u' = -2A\sin 2x + 2B\cos 2x$$
$$u'' = -4A\cos 2x - 4B\sin 2x.$$

Hence,

$$u'' + 4u = -\cancel{4A\cos 2x} - \cancel{4B\sin 2x} + 4(\cancel{A\cos 2x} + \cancel{B\sin 2x}) = 0,$$

which means this is the correct solution.