Exercise 7

Find the general solution for the following initial value problems:

$$u'' - 2u' + 2u = 0, \quad u(0) = 1, \ u'(0) = 1$$

Solution

This is a homogeneous linear ODE with constant coefficients, so the solution is of the form, $u = e^{rx}$.

$$u = e^{rx} \quad \to \quad u' = re^{rx} \quad \to \quad u'' = r^2 e^{rx}$$

Substituting these into the equation gives us

$$r^2 e^{rx} - 2re^{rx} + 2e^{rx} = 0.$$

Divide both sides by e^{rx}.

$$r^2 - 2r + 2 = 0$$

Use the quadratic formula to solve for r.

$$r = \frac{2 \pm \sqrt{4-8}}{2} = 1 \pm i$$

$r = 1 - i$ or $r = 1 + i$, so the general solution is

$$u(x) = C_1 e^{(1-i)x} + C_2 e^{(1+i)x} = e^x (C_1 e^{-ix} + C_2 e^{ix}).$$

But this can be written in terms of sine and cosine by using Euler's formula. Therefore, the general solution is

$$u(x) = e^x (A \cos x + B \sin x).$$

Because we have two initial conditions, we can determine A and B.

$$u'(x) = e^x [(A + B)\cos x + (-A + B)\sin x]$$

$$\begin{aligned} u(0) &= e^0(A) = 1 & &\to \quad A = 1 \\ u'(0) &= e^0(A + B) = 1 & &\to \quad B = 0 \end{aligned}$$

Therefore,

$$u(x) = e^x \cos x.$$

We can check that this is the solution. The first and second derivatives are

$$\begin{aligned} u' &= e^x (\cos x - \sin x) \\ u'' &= -2e^x \sin x. \end{aligned}$$

Hence,

$$u'' - 2u' + 2u = -\cancel{2e^x \sin x} - 2e^x (\cancel{\cos x} - \cancel{\sin x}) + \cancel{2e^x \cos x} = 0,$$

which means this is the correct solution.