Exercise 7

Find the series solution for the following inhomogeneous second order ODEs:

$$u'' - xu = \cos x$$

Solution

Because $x = 0$ is an ordinary point, the series solution of this differential equation will be of the form,

$$u(x) = \sum_{n=0}^{\infty} a_n x^n.$$

To determine the coefficients, a_n, we will have to plug the form into the ODE. Before we can do so, though, we must write expressions for u' and u''.

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad u'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad u''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Also, the Taylor series of $\cos x$ about $x = 0$ is

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}.$$

Now we substitute these series into the ODE.

$$u'' - xu = \cos x$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x \sum_{n=0}^{\infty} a_n x^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

The first series on the left is zero for $n = 0$ and $n = 1$, so we can start the sum from $n = 2$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

Since we want to combine the series on the left, we want the first series to start from $n = 0$. We can start the first at $n = 0$ as long as we replace n with $n + 2$.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$$

To get x^{n+1} in the first series, write out the first term and change n to $n + 1$. Do the same for the series on the right side.

$$2a_2 + \sum_{n=0}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 1 + \sum_{n=0}^{\infty} \frac{(-1)^{n+1}}{(2n+2)!} x^{2n+2}$$

The point of doing this is so that x^{n+1} is present in each term so we can combine the series.

$$2a_2 + \sum_{n=0}^{\infty}[(n+3)(n+2)a_{n+3}x^{n+1} - a_n x^{n+1}] = 1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+2)!}x^{2n+2}$$

Factor the left side.

$$2a_2 + \sum_{n=0}^{\infty}[(n+3)(n+2)a_{n+3} - a_n]x^{n+1} = 1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+2)!}x^{2n+2}$$

We can split the series on the left into two: one for when n is even ($n = 2k$) and another for when n is odd ($n = 2k+1$).

$$\begin{aligned} 2a_2 + \sum_{k=0}^{\infty}[(2k+3)(2k+2)a_{2k+3} - a_{2k}]x^{2k+1} \\ + \sum_{k=0}^{\infty}[(2k+4)(2k+3)a_{2k+4} - a_{2k+1}]x^{2k+2} = 1 + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}}{(2n+2)!}x^{2n+2} \end{aligned}$$

Note that k and n are just dummy indices, so we can put $n = k$ on the right side. Now we match coefficients on both sides.

$$\begin{aligned} &2a_2 = 1 \\ &(2k+3)(2k+2)a_{2k+3} - a_{2k} = 0 \\ &(2k+4)(2k+3)a_{2k+4} - a_{2k+1} = \frac{(-1)^{n+1}}{(2n+2)!} \end{aligned}$$

Now that we know the recurrence relations, we can determine a_n.

$$\begin{aligned} & & 2a_2 = 1 &\quad\rightarrow\quad a_2 = \frac{1}{2} \\ n = 0: & & -a_0 + 6a_3 = 0 &\quad\rightarrow\quad a_3 = \frac{1}{6}a_0 \\ n = 1: & & -a_1 + 12a_4 = -\frac{1}{2} &\quad\rightarrow\quad a_4 = \frac{1}{24}(-1 + 2a_1) \\ n = 2: & & -a_2 + 20a_5 = 0 &\quad\rightarrow\quad a_5 = \frac{1}{40} \\ n = 3: & & -a_3 + 30a_6 = \frac{1}{24} &\quad\rightarrow\quad a_6 = \frac{1}{720}(1 + 4a_0) \\ n = 4: & & -a_4 + 42a_7 = 0 &\quad\rightarrow\quad a_7 = \frac{1}{1008}(-1 + 2a_1) \\ n = 5: & & -a_5 + 56a_8 = -\frac{1}{720} &\quad\rightarrow\quad a_8 = \frac{17}{40320} \\ & & &\quad\vdots\qquad\vdots \end{aligned}$$

Therefore,

$$\begin{aligned} u(x) = a_0\left(1 + \frac{1}{6}x^3 + \frac{1}{180}x^6 + \cdots\right) + a_1\left(x + \frac{1}{12}x^4 + \frac{1}{504}x^7 + \cdots\right) \\ + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{40}x^5 + \frac{1}{720}x^6 - \frac{1}{1008}x^7 + \frac{17}{40320}x^8 + \cdots, \end{aligned}$$

where a_0 and a_1 are arbitrary constants.

[TYPO: $(1/40)x^5$ is repeated in the answer at the back of the book.]