Exercise 8

Find the series solution for the following inhomogeneous second order ODEs:

$$u'' - x^2 u = \ln(1-x)$$

Solution

Because $x = 0$ is an ordinary point, the series solution of this differential equation will be of the form,

$$u(x) = \sum_{n=0}^{\infty} a_n x^n.$$

To determine the coefficients, a_n, we will have to plug the form into the ODE. Before we can do so, though, we must write expressions for u' and u''.

$$u(x) = \sum_{n=0}^{\infty} a_n x^n \quad \rightarrow \quad u'(x) = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \rightarrow \quad u''(x) = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}$$

Also, the Taylor series of $\ln(1-x)$ about $x = 0$ is

$$\ln(1-x) = -\int \frac{1}{1-x}\,dx = -\int \sum_{n=0}^{\infty} x^n\,dx = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}.$$

Now we substitute these series into the ODE.

$$u'' - x^2 u = \ln(1-x)$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - x^2 \sum_{n=0}^{\infty} a_n x^n = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

The first series on the left is zero for $n = 0$ and $n = 1$, so we can start the sum from $n = 2$.

$$\sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} - \sum_{n=0}^{\infty} a_n x^{n+2} = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

Since we want to combine the series on the left, we want the first series to start from $n = 0$. We can start the first at $n = 0$ as long as we replace n with $n + 2$.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} x^n - \sum_{n=0}^{\infty} a_n x^{n+2} = -\sum_{n=0}^{\infty} \frac{1}{n+1} x^{n+1}$$

To get x^{n+2} in the first series, write out the first two terms and change n to $n + 2$. For the series on the right side, write out the first term and change n to $n + 1$.

$$2a_2 + 6a_3 x + \sum_{n=0}^{\infty} (n+4)(n+3) a_{n+4} x^{n+2} - \sum_{n=0}^{\infty} a_n x^{n+2} = -x - \sum_{n=0}^{\infty} \frac{1}{n+2} x^{n+2}$$

The point of doing this is so that x^{n+2} is present in each series so we can combine them.

$$2a_2 + 6a_3x + \sum_{n=0}^{\infty}[(n+4)(n+3)a_{n+4}x^{n+2} - a_nx^{n+2}] = -x - \sum_{n=0}^{\infty}\frac{1}{n+2}x^{n+2}$$

Factor the left side.

$$2a_2 + 6a_3x + \sum_{n=0}^{\infty}[(n+4)(n+3)a_{n+4} - a_n]x^{n+2} = -x - \sum_{n=0}^{\infty}\frac{1}{n+2}x^{n+2}$$

Now we match coefficients on both sides.

$$\begin{aligned} 2a_2 &= 0 \\ 6a_3 &= -1 \\ (n+4)(n+3)a_{n+4} - a_n &= -\frac{1}{n+2} \end{aligned}$$

Now that we know the recurrence relations, we can determine a_n.

$$\begin{array}{rrcl} & 2a_2 = 0 & \rightarrow & a_2 = 0 \\ & 6a_3 = -1 & \rightarrow & a_3 = -\frac{1}{6} \\ n=0: & 12a_4 - a_0 = -\frac{1}{2} & \rightarrow & a_4 = \frac{1}{24}(-1+2a_0) \\ n=1: & 20a_5 - a_1 = -\frac{1}{3} & \rightarrow & a_5 = \frac{1}{60}(-1+3a_1) \\ n=2: & 30a_6 - a_2 = -\frac{1}{4} & \rightarrow & a_6 = -\frac{1}{120} \\ n=3: & 42a_7 - a_3 = -\frac{1}{5} & \rightarrow & a_7 = -\frac{11}{1260} \\ n=4: & 56a_8 - a_4 = -\frac{1}{6} & \rightarrow & a_8 = \frac{1}{1344}(-5+2a_0) \\ n=5: & 72a_9 - a_5 = -\frac{1}{7} & \rightarrow & a_9 = \frac{1}{30240}(-67+21a_1) \\ & & \vdots & \vdots \end{array}$$

Therefore,

$$u(x) = a_0\left(1 + \frac{1}{12}x^4 + \frac{1}{672}x^8 + \cdots\right) + a_1\left(x + \frac{1}{20}x^5 + \frac{1}{1440}x^9 + \cdots\right) - \frac{1}{6}x^3 - \frac{1}{24}x^4 - \frac{1}{60}x^5 - \frac{1}{120}x^6 - \frac{11}{1260}x^7 - \frac{5}{1344}x^8 - \frac{67}{30240}x^9 - \cdots,$$

where a_0 and a_1 are arbitrary constants.