Exercise 15

Differentiate the following $F(x)$ as many times as you need to get rid of the integral sign:

$$F(x) = 1 + \int_0^x (x-t)^3 u(t)\, dt$$

Solution

Take the derivative of both sides with respect to x and use the Leibnitz rule on the integral.

$$F'(x) = 0 + 0 \cdot 1 - x^3 u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x}(x-t)^3 u(t)\, dt$$

The first derivative of $F(x)$ is thus

$$F'(x) = \int_0^x 3(x-t)^2 u(t)\, dt.$$

Differentiate both sides once more with respect to x, again using the Leibnitz rule.

$$F''(x) = 0 \cdot 1 - 3x^2 u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x} 3(x-t)^2 u(t)\, dt$$

The second derivative of $F(x)$ is thus

$$F''(x) = \int_0^x 6(x-t) u(t)\, dt.$$

Differentiate both sides once more with respect to x, again using the Leibnitz rule.

$$F'''(x) = 0 \cdot 1 - 6x u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x} 6(x-t) u(t)\, dt$$

The third derivative of $F(x)$ is thus

$$F'''(x) = \int_0^x 6u(t)\, dt.$$

Differentiate both sides once more with respect to x.

$$F^{(4)}(x) = \frac{d}{dx} \int_0^x 6u(t)\, dt$$

The fundamental theorem of calculus can be applied here to eliminate the integral sign. The fourth derivative of $F(x)$ is thus

$$F^{(4)}(x) = 6u(x).$$