Exercise 16

Differentiate the following $F(x)$ as many times as you need to get rid of the integral sign:

$$F(x) = e^x + \int_0^x (x-t)^4 u(t)\, dt$$

Solution

Take the derivative of both sides with respect to x and use the Leibnitz rule on the integral.

$$F'(x) = e^x + 0 \cdot 1 - x^4 u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x}(x-t)^4 u(t)\, dt$$

The first derivative of $F(x)$ is thus

$$F'(x) = e^x + \int_0^x 4(x-t)^3 u(t)\, dt.$$

Differentiate both sides once more with respect to x, again using the Leibnitz rule.

$$F''(x) = e^x + 0 \cdot 1 - 4x^3 u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x} 4(x-t)^3 u(t)\, dt$$

The second derivative of $F(x)$ is thus

$$F''(x) = e^x + \int_0^x 12(x-t)^2 u(t)\, dt.$$

Differentiate both sides once more with respect to x, again using the Leibnitz rule.

$$F'''(x) = e^x + 0 \cdot 1 - 12x^2 u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x} 12(x-t)^2 u(t)\, dt$$

The third derivative of $F(x)$ is thus

$$F'''(x) = e^x + \int_0^x 24(x-t) u(t)\, dt.$$

Differentiate both sides once more with respect to x, again using the Leibnitz rule.

$$F^{(4)}(x) = e^x + 0 \cdot 1 - 24x u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x} 24(x-t) u(t)\, dt$$

The fourth derivative of $F(x)$ is thus

$$F^{(4)}(x) = e^x + \int_0^x 24 u(t)\, dt.$$

Differentiate both sides once more with respect to x.

$$F^{(5)}(x) = e^x + \frac{d}{dx} \int_0^x 24 u(t)\, dt$$

The fundamental theorem of calculus can be applied here to eliminate the integral sign. The fifth derivative of $F(x)$ is thus

$$F^{(5)}(x) = e^x + 24u(x).$$