Exercise 6

Find $F'(x)$ for the following integrals:

$$F(x) = \int_0^x (x-t)^2 u(t)\, dt$$

Solution

The Leibnitz rule states that if

$$F(x) = \int_{g(x)}^{h(x)} f(x,t)\, dt,$$

then

$$F'(x) = f(x, h(x))\frac{dh}{dx} - f(x, g(x))\frac{dg}{dx} + \int_{g(x)}^{h(x)} \frac{\partial f}{\partial t}\, dt,$$

provided that f and $\partial f/\partial t$ are continuous. In this exercise, $g(x) = 0$, $h(x) = x$, and $f(x,t) = (x-t)^2 u(t)$. Applying the rule gives us

$$F'(x) = 0 \cdot 1 - x^2 u(0) \cdot 0 + \int_0^x \frac{\partial}{\partial x}(x-t)^2 u(t)\, dt.$$

Therefore,

$$F'(x) = \int_0^x 2(x-t)u(t)\, dt.$$