

Exercise 2

Prove the following:

$$\int_0^x \int_0^{x_1} (x-t)^4 u(x_1) dt dx_1 = \frac{1}{5} \int_0^x (x-t)^5 u(t) dt$$

[**TYPO: The integrand should be $(x_1 - t)^4 u(t)$.**]

Solution

From Right to Left

Let

$$G(x) = \frac{1}{5} \int_0^x (x-t)^5 u(t) dt.$$

Note that $G(0) = 0$. Differentiate both sides with respect to x and use the Leibnitz integration rule.

$$\begin{aligned} G'(x) &= \frac{1}{5} \frac{d}{dx} \int_0^x (x-t)^5 u(t) dt \\ &= \frac{1}{5} \int_0^x \frac{\partial}{\partial x} (x-t)^5 u(t) dt + \frac{1}{5} (0)^5 u(x) \cdot 1 - \frac{1}{5} x^5 u(0) \cdot 0 \\ &= \int_0^x (x-t)^4 u(t) dt \end{aligned}$$

Now integrate both sides with respect to x .

$$G(x) = \int_0^x \int_0^{x_1} (x_1-t)^4 u(t) dt dx_1 + C$$

Set the constant of integration and the lower limit of integration to 0 in order to satisfy $G(0) = 0$.

$$G(x) = \int_0^x \int_0^{x_1} (x_1-t)^4 u(t) dt dx_1$$

Therefore,

$$\int_0^x \int_0^{x_1} (x_1-t)^4 u(t) dt dx_1 = \frac{1}{5} \int_0^x (x-t)^5 u(t) dt.$$

From Left to Right

$$\int_0^x \int_0^{x_1} (x_1 - t)^4 u(t) dt dx_1$$

In order to evaluate this double integral, it's necessary to switch the order of integration because $u(t)$ is not given.

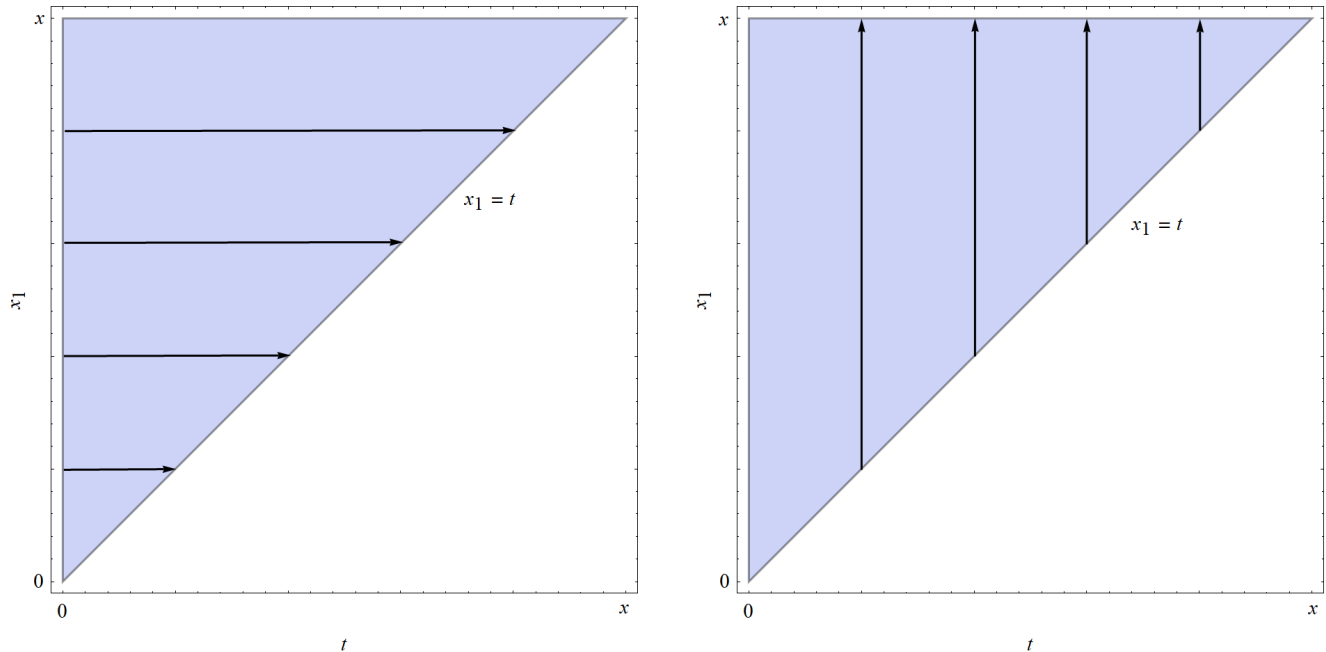


Figure 1: The current mode of integration in the tx_1 -plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$\begin{aligned} \int_0^x \int_0^{x_1} (x_1 - t)^4 u(t) dt dx_1 &= \int_0^x \int_t^x (x_1 - t)^4 u(t) dx_1 dt \\ &= \int_0^x \left. \frac{(x_1 - t)^5}{5} \right|_t^x u(t) dt \\ &= \int_0^x \left[\frac{(x - t)^5}{5} - \frac{0^5}{5} \right] u(t) dt \\ &= \int_0^x \frac{(x - t)^5}{5} u(t) dt \end{aligned}$$

Therefore,

$$\int_0^x \int_0^{x_1} (x_1 - t)^4 u(t) dt dx_1 = \frac{1}{5} \int_0^x (x - t)^5 u(t) dt.$$