Exercise 4

Prove the following:

$$\int_0^x \int_0^{x_1} u(x_1)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x-t)u(x_1)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x-t)^3 u(x_1)\,dt\,dx_1$$
$$= \frac{1}{4}\int_0^x (x-t)^2\left(4+2(x-t)+(x-t)^3\right)u(t)\,dt$$

[TYPO: The integrands should be $u(t)$, $(x_1-t)u(t)$, and $(x_1-t)^3u(t)$, respectively. Also, only one $(x-t)$ should be factored in the integrand on the right side.]

Solution

From Right to Left

Let

$$G(x) = \frac{1}{4}\int_0^x (x-t)\left[4+2(x-t)+(x-t)^3\right]u(t)\,dt.$$

Note that $G(0)=0$. Differentiate both sides with respect to x and use the Leibnitz integration rule.

$$\begin{aligned}
G'(x) &= \frac{1}{4}\frac{d}{dx}\int_0^x (x-t)\left[4+2(x-t)+(x-t)^3\right]u(t)\,dt \\
&= \frac{1}{4}\int_0^x \frac{\partial}{\partial x}(x-t)\left[4+2(x-t)+(x-t)^3\right]u(t)\,dt + \frac{1}{4}(0)[4+2(0)+(0)^3]u(x)\cdot 1 \\
&\qquad - \frac{1}{4}(x)[4+2(x)+(x)^3]u(0)\cdot 0 \\
&= \frac{1}{4}\int_0^x \left\{(1)\left[4+2(x-t)+(x-t)^3\right]+(x-t)\left[2+3(x-t)^2\right]\right\}u(t)\,dt \\
&= \frac{1}{4}\int_0^x [4+4(x-t)+4(x-t)^3]u(t)\,dt \\
&= \int_0^x [1+(x-t)+(x-t)^3]u(t)\,dt
\end{aligned}$$

Now integrate both sides with respect to x.

$$G(x) = \int^x \int_0^{x_1} [1+(x_1-t)+(x_1-t)^3]u(t)\,dt\,dx_1 + C$$

Set the constant of integration and the lower limit of integration to 0 in order to satisfy $G(0)=0$.

$$\begin{aligned}
G(x) &= \int_0^x \int_0^{x_1} [1+(x_1-t)+(x_1-t)^3]u(t)\,dt\,dx_1 \\
&= \int_0^x \int_0^{x_1} u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1-t)u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1-t)^3u(t)\,dt\,dx_1
\end{aligned}$$

Therefore,

$$\int_0^x \int_0^{x_1} u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1-t)u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1-t)^3u(t)\,dt\,dx_1$$
$$= \frac{1}{4}\int_0^x (x-t)[4+2(x-t)+(x-t)^3]u(t)\,dt.$$

From Left to Right

Combine the integrals.

$$\int_0^x \int_0^{x_1} u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1 - t)u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1 - t)^3 u(t)\,dt\,dx_1$$
$$= \int_0^x \int_0^{x_1} [1 + (x_1 - t) + (x_1 - t)^3]u(t)\,dt\,dx_1$$

In order to evaluate this double integral, it's necessary to switch the order of integration because $u(t)$ is not given.

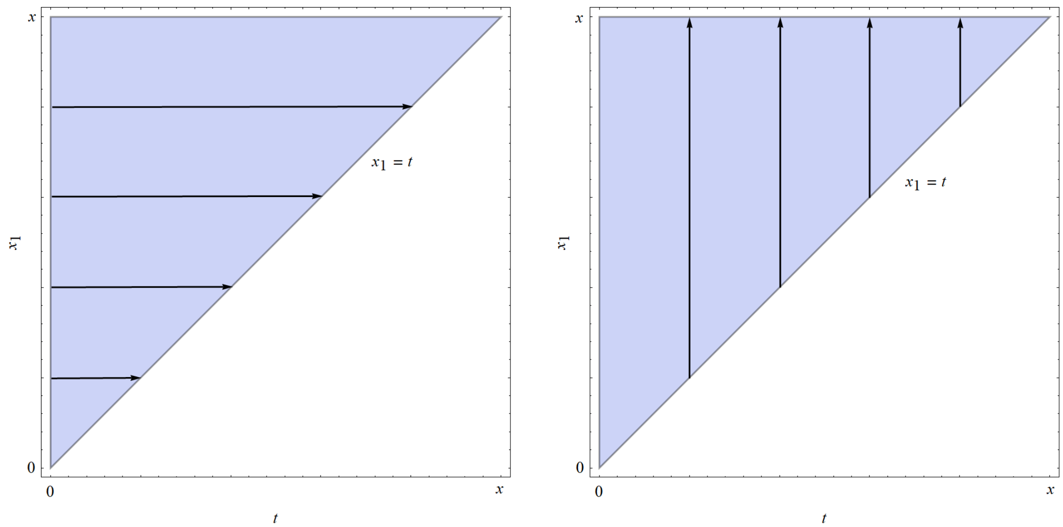

Figure 1: The current mode of integration in the tx_1-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$\begin{aligned}\int_0^x \int_0^{x_1} [1 + (x_1 - t) + (x_1 - t)^3]u(t)\,dt\,dx_1 &= \int_0^x \int_t^x [1 + (x_1 - t) + (x_1 - t)^3]u(t)\,dx_1\,dt \\ &= \int_0^x \left[x_1 + \frac{(x_1 - t)^2}{2} + \frac{(x_1 - t)^4}{4}\right]\Bigg|_t^x u(t)\,dt \\ &= \int_0^x \left[(x - t) + \frac{(x - t)^2}{2} + \frac{(x - t)^4}{4}\right] u(t)\,dt \\ &= \frac{1}{4}\int_0^x [4(x - t) + 2(x - t)^2 + (x - t)^4]u(t)\,dt \\ &= \frac{1}{4}\int_0^x (x - t)[4 + 2(x - t) + (x - t)^3]u(t)\,dt\end{aligned}$$

Therefore,

$$\int_0^x \int_0^{x_1} u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1 - t)u(t)\,dt\,dx_1 + \int_0^x \int_0^{x_1} (x_1 - t)^3 u(t)\,dt\,dx_1$$
$$= \frac{1}{4}\int_0^x (x - t)[4 + 2(x - t) + (x - t)^3]u(t)\,dt.$$