Exercise 10

Find the Laplace transform of the following expressions that include convolution products:

$$x^2 + \int_0^x e^{x-t} y(t)\, dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\, dx.$$

Take the Laplace transform of the provided expression.

$$\begin{aligned}
\mathcal{L}\left\{x^2 + \int_0^x e^{x-t} y(t)\, dt\right\} &= \int_0^\infty e^{-sx}\left[x^2 + \int_0^x e^{x-t} y(t)\, dt\right] dx \\
&= \int_0^\infty x^2 e^{-sx}\, dx + \int_0^\infty e^{-sx} \int_0^x e^{x-t} y(t)\, dt\, dx \\
&= \int_0^\infty \frac{\partial^2}{\partial s^2}(e^{-sx})\, dx + \int_0^\infty \int_0^x e^{-sx} e^{x-t} y(t)\, dt\, dx
\end{aligned}$$

In order to evaluate the double integral, the order of integration has to be switched.

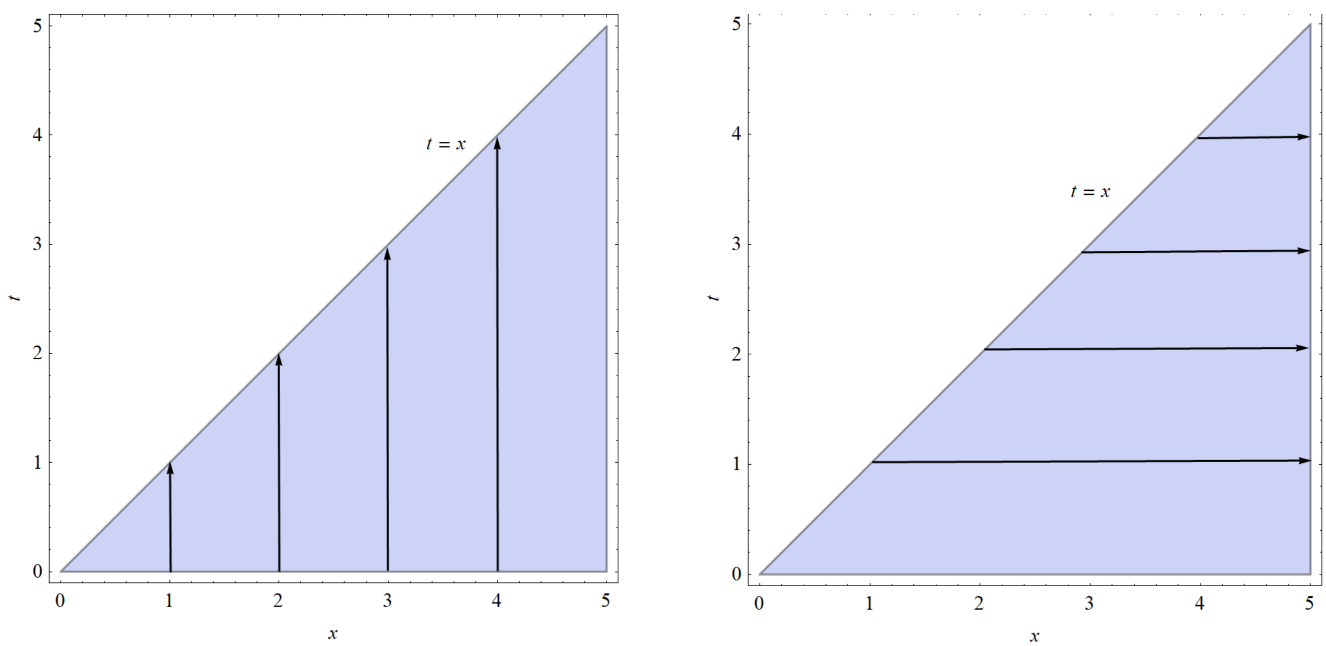

Figure 1: The current mode of integration in the xt-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the double integral.

$$\mathcal{L}\left\{x^2 + \int_0^x e^{x-t} y(t)\, dt\right\} = \frac{d^2}{ds^2}\int_0^\infty e^{-sx}\, dx + \int_0^\infty \int_t^\infty e^{-sx} e^{x-t} y(t)\, dx\, dt$$

Now make the following substitution.

$$\begin{aligned} r &= x - t \quad \rightarrow \quad r + t = x \\ dr &= dx \end{aligned}$$

The double integral can then be evaluated.

$$\begin{aligned}
\mathcal{L}\left\{x^2 + \int_0^x e^{x-t} y(t)\,dt\right\} &= \frac{d^2}{ds^2}\left[\frac{1}{(-s)}e^{-sx}\bigg|_0^\infty\right] + \int_0^\infty \int_0^\infty e^{-s(r+t)} e^r y(t)\,dr\,dt \\
&= \frac{d^2}{ds^2}\left(\frac{1}{s}\right) + \int_0^\infty \int_0^\infty e^{-sr} e^{-st} e^r y(t)\,dr\,dt \\
&= \frac{2}{s^3} + \left[\int_0^\infty e^{-sr} e^r\,dr\right]\left[\int_0^\infty e^{-st} y(t)\,dt\right] = \mathcal{L}\{x^2\} + \mathcal{L}\{e^x\}\mathcal{L}\{y(x)\} \\
&= \frac{2}{s^3} + \left[\int_0^\infty e^{(-s+1)r}\,dr\right] Y(s) \\
&= \frac{2}{s^3} + \left[\frac{1}{-s+1} e^{(-s+1)r}\bigg|_0^\infty\right] Y(s) \\
&= \frac{2}{s^3} + \left(\frac{1}{s-1}\right) Y(s)
\end{aligned}$$

Therefore,

$$\mathcal{L}\left\{x^2 + \int_0^x e^{x-t} y(t)\,dt\right\} = \frac{2}{s^3} + \frac{Y(s)}{s-1}.$$