Exercise 8

Find the Laplace transform of the following expressions:

$$\sin x + \sinh x$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx.$$

Use the definition to find the answer.

$$\begin{aligned}
\mathcal{L}\{\sin x + \sinh x\} &= \int_0^\infty e^{-sx}(\sin x + \sinh x)\,dx \\
&= \int_0^\infty (e^{-sx}\sin x + e^{-sx}\sinh x)\,dx \\
&= \int_0^\infty e^{-sx}\sin x\,dx + \int_0^\infty e^{-sx}\sinh x\,dx
\end{aligned}$$

Express both $\sin x$ and $\sinh x$ in terms of exponential functions.

$$\begin{aligned}
&= \int_0^\infty e^{-sx}\left(\frac{e^{ix} - e^{-ix}}{2i}\right)dx + \int_0^\infty e^{-sx}\left(\frac{e^{x} - e^{-x}}{2}\right)dx \\
&= \frac{1}{2i}\left[\int_0^\infty e^{(-s+i)x}\,dx - \int_0^\infty e^{-(s+i)x}\,dx\right] + \frac{1}{2}\left[\int_0^\infty e^{(-s+1)x}\,dx - \int_0^\infty e^{-(s+1)x}\,dx\right] \\
&= \frac{1}{2i}\left[\frac{1}{-s+i}e^{(-s+i)x}\bigg|_0^\infty - \frac{1}{-(s+i)}e^{-(s+i)x}\bigg|_0^\infty\right] \\
&\qquad + \frac{1}{2}\left[\frac{1}{-s+1}e^{(-s+1)x}\bigg|_0^\infty - \frac{1}{-(s+1)}e^{-(s+1)x}\bigg|_0^\infty\right] \\
&= \frac{1}{2i}\left(\frac{1}{s-i} - \frac{1}{s+i}\right) + \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) \\
&= \frac{1}{2i}\cdot\frac{s+i-s+i}{(s-i)(s+i)} + \frac{1}{2}\cdot\frac{s+1-s+1}{(s-1)(s+1)} \\
&= \frac{1}{\cancel{2i}}\cdot\frac{\cancel{2i}}{s^2-i^2} + \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{s^2-1} \\
&= \frac{1}{s^2+1} + \frac{1}{s^2-1} \\
&= \frac{s^2-1+s^2+1}{(s^2+1)(s^2-1)}
\end{aligned}$$

Therefore,

$$\mathcal{L}\{\sin x + \sinh x\} = \frac{2s^2}{s^4-1}.$$