Exercise 9

Find the Laplace transform of the following expressions that include convolution products:

$$\int_0^x \sinh(x-t)y(t)\,dt$$

Solution

The Laplace transform of a function $f(x)$ is defined as

$$F(s) = \mathcal{L}\{f(x)\} = \int_0^\infty e^{-sx} f(x)\,dx.$$

Take the Laplace transform of the provided expression.

$$\begin{aligned}\mathcal{L}\left\{\int_0^x \sinh(x-t)y(t)\,dt\right\} &= \int_0^\infty e^{-sx} \int_0^x \sinh(x-t)y(t)\,dt\,dx \\ &= \int_0^\infty \int_0^x e^{-sx}\sinh(x-t)y(t)\,dt\,dx\end{aligned}$$

The aim is to make a substitution so that sinh and y are both functions of only one variable. Since the inner integral is in dt and t is present in both sinh and y, a substitution won't achieve anything. x is only present in sinh, so if we change the order of integration to make the inner integral in dx, then a substitution will lead to progress.

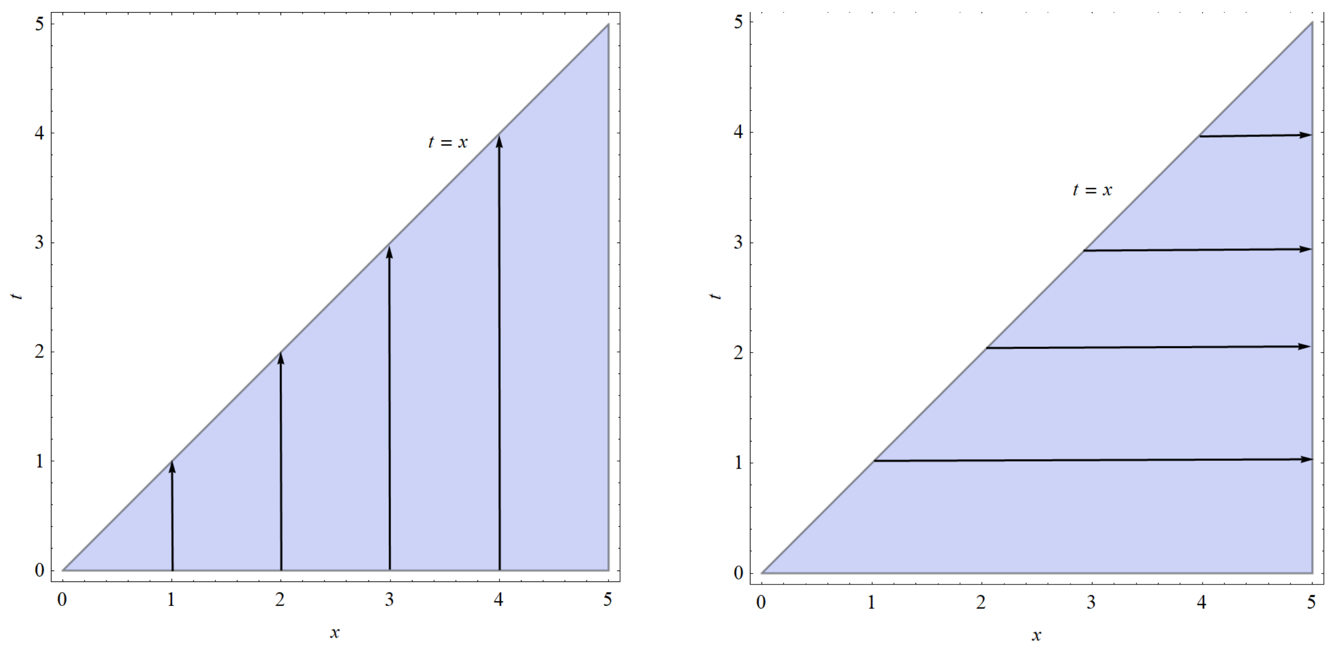

Figure 1: The current mode of integration in the xt-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$\mathcal{L}\left\{\int_0^x \sinh(x-t)y(t)\,dt\right\} = \int_0^\infty \int_t^\infty e^{-sx}\sinh(x-t)y(t)\,dx\,dt$$

Now make the following substitution.

$$\begin{aligned} r &= x - t \quad \rightarrow \quad r + t = x \\ dr &= dx \end{aligned}$$

The double integral can then be evaluated.

$$\begin{aligned} \mathcal{L}\left\{\int_0^x \sinh(x-t)y(t)\,dt\right\} &= \int_0^\infty \int_0^\infty e^{-s(r+t)} \sinh(r) y(t)\,dr\,dt \\ &= \int_0^\infty \int_0^\infty e^{-sr} e^{-st} \sinh(r) y(t)\,dr\,dt \\ &= \left[\int_0^\infty e^{-sr} \sinh(r)\,dr\right] \left[\int_0^\infty e^{-st} y(t)\,dt\right] = \mathcal{L}\{\sinh x\}\mathcal{L}\{y(x)\} \\ &= \left[\int_0^\infty e^{-sr} \left(\frac{e^r - e^{-r}}{2}\right) dr\right] Y(s) \\ &= \frac{1}{2}\left[\int_0^\infty e^{(-s+1)r}\,dr - \int_0^\infty e^{-(s+1)r}\,dr\right] Y(s) \\ &= \frac{1}{2}\left[\frac{1}{-s+1} e^{(-s+1)r}\bigg|_0^\infty - \frac{1}{-(s+1)} e^{-(s+1)r}\bigg|_0^\infty\right] Y(s) \\ &= \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right) Y(s) \\ &= \frac{1}{2} \cdot \frac{s+1-s+1}{(s-1)(s+1)} Y(s) \\ &= \frac{1}{\cancel{2}} \cdot \frac{\cancel{2}}{s^2-1} Y(s) \end{aligned}$$

Therefore,

$$\mathcal{L}\left\{\int_0^x \sinh(x-t)y(t)\,dt\right\} = \frac{Y(s)}{s^2-1}.$$