Exercise 10

Show that

$$\int_2^{e+1} \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots \right) dx = 1, \ x > 1$$

Solution

Inspecting the series in the parentheses, we see that it is geometric. The first term is

$$a_1 = \frac{1}{x},$$

and the common ratio is

$$r = \frac{1}{x} < 1.$$

Consequently, the sum of the series is

$$\begin{aligned} S &= \frac{a_1}{1-r} \\ &= \frac{\frac{1}{x}}{1-\frac{1}{x}} \\ &= \frac{1}{x-1}. \end{aligned}$$

The integral that we have to evaluate then is

$$\begin{aligned} \int_2^{e+1} \frac{dx}{x-1} &= \ln(x-1)\bigg|_2^{e+1} \\ &= \ln e - \ln 1 \\ &= 1. \end{aligned}$$

Therefore,

$$\int_2^{e+1} \left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \cdots \right) dx = 1, \ x > 1.$$