Exercise 15

Convert each of the following Volterra integral equation in 9–16 to an equivalent IVP:

$$u(x) = 1 + 2\int_0^x (x-t)^3 u(t)\,dt$$

Solution

Differentiate both sides with respect to x.

$$u'(x) = 2\frac{d}{dx}\int_0^x (x-t)^3 u(t)\,dt$$

Use the Leibnitz rule to differentiate the integral.

$$\begin{aligned}
&= 2\left[\int_0^x \frac{\partial}{\partial x}(x-t)^3 u(t)\,dt + (0)^3 u(x)\cdot 1 - (x)^3 u(0)\cdot 0\right] \\
&= 2\left[\int_0^x 3(x-t)^2 u(t)\,dt\right] \\
&= 6\int_0^x (x-t)^2 u(t)\,dt
\end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned}
u''(x) &= 6\frac{d}{dx}\int_0^x (x-t)^2 u(t)\,dt \\
&= 6\left[\int_0^x \frac{\partial}{\partial x}(x-t)^2 u(t)\,dt + (0)^2 u(x)\cdot 1 - (x)^2 u(0)\cdot 0\right] \\
&= 6\left[\int_0^x 2(x-t) u(t)\,dt\right] \\
&= 12\int_0^x (x-t) u(t)\,dt
\end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned}
u'''(x) &= 12\frac{d}{dx}\int_0^x (x-t) u(t)\,dt \\
&= 12\left[\int_0^x \frac{\partial}{\partial x}(x-t) u(t)\,dt + (0) u(x)\cdot 1 - (x) u(0)\cdot 0\right] \\
&= 12\int_0^x u(t)\,dt
\end{aligned}$$

Differentiate both sides with respect to x again.

$$\begin{aligned}
u^{(\text{iv})}(x) &= 12\frac{d}{dx}\int_0^x u(t)\,dt \\
&= 12u(x)
\end{aligned}$$

$$u^{(\text{iv})} - 12u = 0$$

The initial conditions to this ODE are found by plugging in $x = 0$ into the original integral equation,

$$u(0) = 1 + 2\int_0^0 (0-t)^3 u(t)\,dt = 1,$$

and the formula for u',

$$u'(0) = 6\int_0^0 (0-t)^2 u(t)\,dt = 0,$$

and the formula for u'',

$$u''(0) = 12\int_0^0 (0-t) u(t)\,dt = 0,$$

and the formula for u''',

$$u'''(0) = 12\int_0^0 u(t)\,dt = 0.$$

Therefore, the equivalent IVP is

$$u^{(\text{iv})} - 12u = 0,\ u(0) = 1,\ u'(0) = 0,\ u''(0) = 0,\ u'''(0) = 0.$$

[TYPO: The answer at the back of the book reads:

$$u^{\text{iv}}(x) - 12u(x) = 0,\ u(0) = 1,\ u'(0) = u''(0) = u''' = 0$$

Parentheses need to be placed around “iv” and (0) needs to be placed after u'''.]