Exercise 7

Convert each of the following IVPs in 1–8 to an equivalent Volterra integral equation:

$$y^{(\text{iv})} - y'' = 1,\ y(0) = y'(0) = 0,\ y''(0) = y'''(0) = 1$$

Solution

Let

$$y^{(\text{iv})}(x) = u(x). \tag{1}$$

Integrate both sides from 0 to x.

$$\begin{aligned} \int_0^x y^{(\text{iv})}(t)\,dt &= \int_0^x u(t)\,dt \\ y'''(x) - y'''(0) &= \int_0^x u(t)\,dt \end{aligned}$$

Substitute $y'''(0) = 1$ and bring it to the right side.

$$y'''(x) = 1 + \int_0^x u(t)\,dt \tag{2}$$

Integrate both sides again from 0 to x.

$$\begin{aligned} \int_0^x y'''(s)\,ds &= \int_0^x \left[1 + \int_0^s u(t)\,dt\right] ds \\ y''(x) - y''(0) &= x + \int_0^x \int_0^s u(t)\,dt\,ds \end{aligned}$$

Substitute $y''(0) = 1$ and bring it to the right side.

$$y''(x) = 1 + x + \int_0^x \int_0^s u(t)\,dt\,ds$$

Use integration by parts to write the double integral as a single integral. Let

$$\begin{aligned} v &= \int_0^s u(t)\,dt & dw &= ds \\ dv &= u(s)\,ds & w &= s \end{aligned}$$

and use the formula $\int v\,dw = vw - \int w\,dv$.

$$\begin{aligned} y''(x) &= 1 + x + s\int_0^s u(t)\,dt\Big|_0^x - \int_0^x su(s)\,ds \\ &= 1 + x + x\int_0^x u(t)\,dt - \int_0^x su(s)\,ds \\ &= 1 + x + x\int_0^x u(t)\,dt - \int_0^x tu(t)\,dt \\ &= 1 + x + \int_0^x (x-t)u(t)\,dt \end{aligned} \tag{3}$$

Integrate both sides again from 0 to x.

$$\int_0^x y''(r)\,dr = \int_0^x \left[1 + r + \int_0^r (r-t)u(t)\,dt\right] dr$$
$$y'(x) - y'(0) = x + \frac{x^2}{2} + \int_0^x \int_0^r (r-t)u(t)\,dt\,dr$$

Substitute $y'(0) = 0$.

$$y'(x) = x + \frac{x^2}{2} + \int_0^x \int_0^r (r-t)u(t)\,dt\,dr$$

In order to evaluate the double integral, switch the order of integration so that dr comes first.

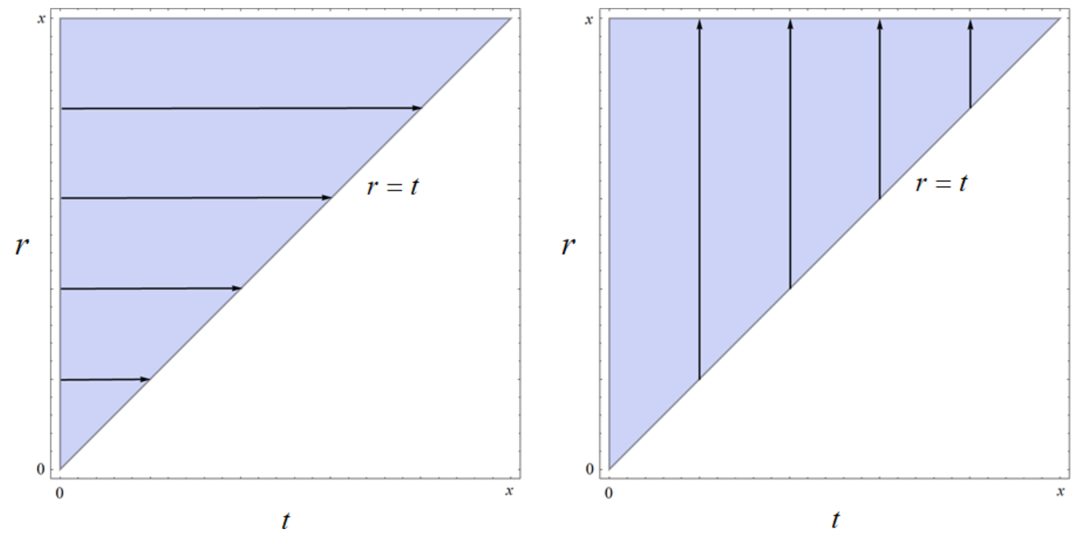

Figure 1: The current mode of integration in the tr-plane is shown on the left. This domain will be integrated over as shown on the right to simplify the integral.

$$\begin{aligned} y'(x) &= x + \frac{x^2}{2} + \int_0^x \int_t^x (r-t)u(t)\,dr\,dt \\ &= x + \frac{x^2}{2} + \int_0^x \left[\frac{(r-t)^2}{2}\right]\Bigg|_t^x u(t)\,dt \\ &= x + \frac{x^2}{2} + \int_0^x \frac{(x-t)^2}{2} u(t)\,dt \\ &= x + \frac{x^2}{2} + \frac{1}{2}\int_0^x (x-t)^2 u(t)\,dt \end{aligned} \tag{4}$$

Integrate both sides again from 0 to x.

$$\int_0^x y'(q)\,dq = \int_0^x \left[q + \frac{q^2}{2} + \frac{1}{2}\int_0^q (q-t)^2 u(t)\,dt\right] dq$$
$$y(x) - y(0) = \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{2}\int_0^x \int_0^q (q-t)^2 u(t)\,dt\,dq$$

Substitute $y(0) = 0$.

$$y(x) = \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{2}\int_0^x \int_0^q (q-t)^2 u(t)\,dt\,dq$$

Switch the order of integration as we did before.

$$\begin{aligned}
&= \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{2}\int_0^x \int_t^x (q-t)^2 u(t)\,dq\,dt \\
&= \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{2}\int_0^x \left[\frac{(q-t)^3}{3}\right]\bigg|_t^x u(t)\,dt \\
&= \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{2}\int_0^x \left[\frac{(x-t)^3}{3}\right] u(t)\,dt \\
&= \frac{x^2}{2} + \frac{x^3}{6} + \frac{1}{6}\int_0^x (x-t)^3 u(t)\,dt
\end{aligned} \tag{5}$$

Substitute equations (1), (2), (3), (4), and (5) into the original ODE.

$$y^{(\text{iv})} - y'' = 1 \quad \rightarrow \quad u(x) - \left[1 + x + \int_0^x (x-t)u(t)\,dt\right] = 1$$

Expand the left side.

$$u(x) - 1 - x - \int_0^x (x-t)u(t)\,dt = 1$$

Therefore, the equivalent Volterra integral equation is

$$u(x) = 2 + x + \int_0^x (x-t)u(t)\,dt.$$