

## Exercise 1

Convert each of the following BVPs in 1–8 to an equivalent Fredholm integral equation:

$$y'' + 4y = 0, \quad 0 < x < 1, \quad y(0) = y(1) = 0$$

### Solution

Let

$$y''(x) = u(x). \tag{1}$$

Integrate both sides from 0 to  $x$ .

$$\begin{aligned} \int_0^x y''(t) dt &= \int_0^x u(t) dt \\ y'(x) - y'(0) &= \int_0^x u(t) dt \end{aligned}$$

Bring  $y'(0)$  to the right side.

$$y'(x) = y'(0) + \int_0^x u(t) dt$$

Integrate both sides from 0 to  $x$  again.

$$\begin{aligned} \int_0^x y'(r) dr &= \int_0^x \left[ y'(0) + \int_0^r u(t) dt \right] dr \\ y(x) - y(0) &= y'(0)x + \int_0^x \int_0^r u(t) dt dr \end{aligned}$$

Substitute  $y(0) = 0$ .

$$y(x) = y'(0)x + \int_0^x \int_0^r u(t) dt dr$$

Use integration by parts to write the double integral as a single integral. Let

$$\begin{aligned} v &= \int_0^r u(t) dt & dw &= dr \\ dv &= u(r) dr & w &= r \end{aligned}$$

and use the formula  $\int v dw = vw - \int w dv$ .

$$\begin{aligned} y(x) &= y'(0)x + r \int_0^r u(t) dt \Big|_0^x - \int_0^x ru(r) dr \\ &= y'(0)x + x \int_0^x u(t) dt - \int_0^x ru(r) dr \\ &= y'(0)x + x \int_0^x u(t) dt - \int_0^x tu(t) dt \\ &= y'(0)x + \int_0^x (x-t)u(t) dt \end{aligned}$$

In order to determine  $y'(0)$ , set  $x = 1$  in this equation for  $y(x)$ .

$$y(1) = y'(0) + \int_0^1 (1-t)u(t) dt$$

Substitute  $y(1) = 0$  and solve for  $y'(0)$ .

$$0 = y'(0) + \int_0^1 (1-t)u(t) dt \quad \rightarrow \quad y'(0) = - \int_0^1 (1-t)u(t) dt$$

Plug this result for  $y'(0)$  back into the formula for  $y(x)$ .

$$y(x) = -x \int_0^1 (1-t)u(t) dt + \int_0^x (x-t)u(t) dt \quad (2)$$

Now plug equations (1) and (2) into the original ODE.

$$y'' + 4y = 0 \quad \rightarrow \quad u(x) + 4 \left[ -x \int_0^1 (1-t)u(t) dt + \int_0^x (x-t)u(t) dt \right] = 0$$

Expand the left side.

$$u(x) - 4x \int_0^1 (1-t)u(t) dt + 4 \int_0^x (x-t)u(t) dt = 0$$

Solve for  $u(x)$ .

$$\begin{aligned} u(x) &= 4x \int_0^1 (1-t)u(t) dt - 4 \int_0^x (x-t)u(t) dt \\ &= \int_0^1 4x(1-t)u(t) dt - \int_0^x 4(x-t)u(t) dt \\ &= \int_0^x 4x(1-t)u(t) dt + \int_x^1 4x(1-t)u(t) dt - \int_0^x 4(x-t)u(t) dt \\ &= \int_0^x [4x(1-t) - 4(x-t)]u(t) dt + \int_x^1 4x(1-t)u(t) dt \\ &= \int_0^x (-4xt + 4t)u(t) dt + \int_x^1 4x(1-t)u(t) dt \\ &= \int_0^x 4t(1-x)u(t) dt + \int_x^1 4x(1-t)u(t) dt \end{aligned}$$

Therefore, the equivalent Fredholm integral equation is

$$u(x) = \int_0^1 K(x,t)u(t) dt,$$

where

$$K(x,t) = \begin{cases} 4t(1-x) & 0 \leq t \leq x \\ 4x(1-t) & x \leq t \leq 1 \end{cases}.$$